

A WALL CROSSING FORMULA FOR DEGREES OF REAL CENTRAL PROJECTIONS

CHRISTIAN OKONEK AND ANDREI TELEMAN

0. INTRODUCTION

One of the fundamental problems in real algebraic geometry concerns the solutions of systems of real algebraic equations. Since, in general, even the existence of real solutions is not guaranteed, it is important to find a priori lower bounds for the number of these solutions.

In the last years several important developments have taken place in this direction, which are related to problems in enumerative geometry [DeK], [IKS], [FK], [W1], [W2], [OT2], or to the study of certain polynomial systems which often have interesting applications [EG1], [EG2], [SS]. In many cases these lower bounds are provided by the degrees of certain maps, for instance central projections of projective submanifolds [EG1], [EG2], [So], [SS]. For a projective m -dimensional submanifold $X \subset \mathbb{P}_{\mathbb{C}}^{N-1}$, and a central projection $\pi : \mathbb{P}_{\mathbb{C}}^{N-1} \dashrightarrow \mathbb{P}_{\mathbb{C}}^m$ whose center does not intersect X , we get a finite map $\pi|_X : X \rightarrow \mathbb{P}_{\mathbb{C}}^m$ whose degree with respect to complex orientations coincides with the number of points of a fibre $(\pi|_X)^{-1}(p)$ if multiplicities are taken into account. Note that this fibre can be regarded as the set of solutions of a system of homogeneous algebraic equations. This degree can also be identified with the number of points of the intersection $X \cap H$ for a general codimension $N - n$ projective subspace $H \subset \mathbb{P}_{\mathbb{C}}^N$; hence it is cohomologically determined and independent of the choice of the central projection π (as long as its center does not intersect X). This is an important example which illustrates the general principle of *conservation of numbers*. It is well known that this principle does not hold in real algebraic geometry. One of the goals of this article is to show that in real geometry, this principle should be replaced by a *wall crossing formula*. Such wall crossing formulae play an important role in gauge theory [DoK] [OT1], but apparently the wall crossing phenomenon (jump of an invariant in a well controlled way) has not been studied until now in real algebraic geometry. The general principle of wall crossing is very simple: one has a parameter space \mathcal{P} which parameterizes a certain class of maps, and a wall $\mathcal{W} \subset \mathcal{P}$ of *bad* maps. If certain conditions are fulfilled, one can define a degree map

$$\deg : \pi_0(\mathcal{P} \setminus \mathcal{W}) \rightarrow \mathbb{Z}$$

which associates to any chamber $C \in \pi_0(\mathcal{P} \setminus \mathcal{W})$ an integer. A wall crossing formula computes the difference $\deg(C_+) - \deg(C_-)$ between the integers associated to two adjacent chambers. In this article we will prove such a wall crossing formula for the space of central projections restricted to a submanifold X of a real projective space, *which is not necessarily algebraic*. The crucial assumption on the class of maps we consider is *relative orientability*, a condition which allows us to define a \mathbb{Z} -valued degree map in a coherent way. This wall crossing phenomenon for real

central projections $\mathbb{P}(V) \supset X \rightarrow \mathbb{P}(W)$ pointed out by our result is in striking contrast to the invariance of the degree of a section in a relatively oriented vector bundle (see [OT2]). We describe now briefly the results of this article:

In the first section we introduce the important notion of relative orientation of a map between topological manifolds, and we define the degree of a relatively oriented map between closed manifolds of the same dimension. For relatively *orientable* maps f (so for maps which can be relatively oriented, but have not been endowed with a relative orientation) one can define the absolute degree $|f|$, which is similar to Kronecker's concept of *characteristic*. In the differentiable case the degree of a relatively oriented map can be computed using the local degrees at the points of a finite fibre. The special case of a Real finite holomorphic map $f : X \rightarrow Y$ between Real complex manifolds of the same dimension is particularly important. If the restriction

$$f(\mathbb{R}) : X(\mathbb{R}) \rightarrow Y(\mathbb{R})$$

is relatively orientable, then there are fundamental estimates and comparison formulae for the sum of the multiplicities of the points of a real fibre $f(\mathbb{R})^{-1}(y)$, $y \in Y(\mathbb{R})$:

$$\begin{aligned} |\deg|(f(\mathbb{R})) &\leq \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) \leq \deg(f) , \\ |\deg|(f(\mathbb{R})) &\equiv \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) \equiv \deg(f) \pmod{2} . \end{aligned}$$

In the second section we study central projections. Let V, W be real vector spaces, $f \in \text{Hom}(V, W)$ and

$$[f] : \mathbb{P}(V) \setminus \mathbb{P}(\ker(f)) \rightarrow \mathbb{P}(W)$$

the induced morphism. Let $X \subset \mathbb{P}(V)$ be a compact submanifold with $\dim(X) = \dim(\mathbb{P}(W))$ and $X \cap \mathbb{P}(\ker(f)) = \emptyset$. The induced map

$$[f]_X : X \rightarrow \mathbb{P}(W)$$

is relatively orientable if and only if

$$w_1(X) = (\dim(X) + 1)w_1(\lambda_{V,X}) ,$$

where $\lambda_{V,X} := \lambda_V|_X$ denotes the restriction to X of the tautological line bundle λ_V of $\mathbb{P}(V)$. It follows in particular that the relative orientability of such central projections $[f]_X$ is independent of f .

The tensor product $T_X \otimes \lambda_{V,X}$ can be written as $Y_X / \lambda_{V,X}$, where Y_X is a subbundle of the trivial bundle $\underline{V}_X := X \times V$. With this definition we obtain a canonical identification

$$T_X = \text{Hom}(\lambda_{V,X}, Y_X / \lambda_{V,X}) .$$

The data of a relative orientation of a map $f \in \text{Hom}(V, W)$ with $X \cap \mathbb{P}(\ker(f)) = \emptyset$ is equivalent to the data of a bundle isomorphism

$$\mu : X \times \det(W) \rightarrow \det(Y_X) .$$

In our situation the space \mathcal{P} parameterizing the relevant maps is the vector space $\text{Hom}(V, W)$, and the wall associated with the submanifold $X \subset \mathbb{P}(V)$ is:

$$\mathcal{W}_X := \{f \in \text{Hom}(V, W) \mid X \cap \mathbb{P}(\ker(f)) \neq \emptyset\} .$$

A point $f_0 \in \mathcal{W}_X$ on the wall is called regular when f_0 is surjective, the intersection $X \cap \mathbb{P}(\ker(f_0))$ consists only of one point ξ_0 , and $\ker(f_0) \cap Y_{\xi_0} = \xi_0$. Let \mathcal{W}_X^0 be the subspace of regular points in \mathcal{W}_X . Our first result in section 2 shows that the wall $\mathcal{W}_X \subset \text{Hom}(V, W)$ is a smooth hypersurface in all regular points $f_0 \in \mathcal{W}_X^0$ and identifies the normal line $N_{\mathcal{W}_X^0, f_0}$ with $\text{Hom}(\det(Y_{\xi_0}), \det(W))$ via a canonical isomorphism

$$\psi_{f_0} : N_{\mathcal{W}_X^0, f_0} \rightarrow \text{Hom}(\det(Y_{\xi_0}), \det(W)) .$$

This shows that the choice of an orientation parameter $\mu : X \times \det(W) \rightarrow \det(Y_X)$ serves a second, completely different purpose: $\mu_{\xi_0}^{-1} : \det(Y_{\xi_0}) \rightarrow \det(W)$ defines a generator of the normal line $N_{\mathcal{W}_X^0, f_0}$ for every $f_0 \in \mathcal{W}_X^0$ with $\ker(f_0) \cap Y_{\xi_0} = \xi_0$.

The main result in this section is the wall crossing formula (see Theorem 20):

Theorem (Wall-crossing formula) *Let $f_0 \in \mathcal{W}_X^0$ be a regular point on the wall with $\ker(f_0) \cap Y_{\xi_0} = \xi_0$. Consider a smooth map*

$$\mathfrak{f} : N_{\mathcal{W}_X^0, f_0} \rightarrow \text{Hom}(V, W)$$

whose differential $\mathfrak{f}_{,0}$ is a right splitting of the exact sequence*

$$0 \rightarrow T_{\mathcal{W}_X^0, f_0} \rightarrow T_{\text{Hom}(V, W), f_0} \rightarrow N_{\mathcal{W}_X^0, f_0} \rightarrow 0 .$$

Then for every sufficiently small $\tau \in N_0 \setminus \{0\}$ we have:

- (1) $\mathfrak{f}(\tau) \in \text{Hom}(V, W) \setminus \mathcal{W}_X$ and $[\mathfrak{f}(\tau)]_X$ is a local diffeomorphism at ξ_0 .
- (2) The local degree of $[\mathfrak{f}(\tau)]_X$ at ξ_0 is

$$\deg_{\nu(\mathfrak{f}(\tau), \mu), \xi_0}([\mathfrak{f}(\tau)]_X) = \text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\tau))) .$$

- (3) $\deg_{\nu(\mathfrak{f}(\tau), \mu)}([\mathfrak{f}(\tau)]_X) - \deg_{\nu(\mathfrak{f}(-\tau), \mu)}([\mathfrak{f}(-\tau)]_X) = 2\text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\tau)))$.

As a consequence we obtain a general formula which computes the degree difference $\deg_{\nu(g_1, \mu)}([g_1]_X) - \deg_{\nu(g_0, \mu)}([g_0]_X)$ for a smooth path $(g_t)_{t \in [0, 1]}$ in $\text{Hom}(V, W)$ with $g_0, g_1 \in \text{Hom}(V, W) \setminus \mathcal{W}_X$, which intersects the wall \mathcal{W}_X only in regular points with transversal intersection.

The third result in section 2 describes the irregular locus $\mathcal{W}_X \setminus \mathcal{W}_X^0$ of the wall. We show that $\mathcal{W}_X \setminus \mathcal{W}_X^0$ is closed and its complement in $\text{Hom}(V, W)$ is connected, so that any two points $g_0, g_1 \in \text{Hom}(V, W) \setminus \mathcal{W}_X$ can be connected by a smooth path $(g_t)_{t \in [0, 1]}$ which intersects \mathcal{W}_X only along \mathcal{W}_X^0 with transversal intersection. In other words, the difference $\deg_{\nu(g_1, \mu)}([g_1]_X) - \deg_{\nu(g_0, \mu)}([g_0]_X)$ can always be computed by our difference formula.

In the third section we discuss examples. First we describe a large class of important Real complex manifolds, the so called conjugation manifolds [HHP], for which the orientability of $f(\mathbb{R})$ can be easily checked. This class contains Grassmann manifolds and toric manifolds with their standard Real structures. Then we discuss Wronski projections, the universal pole placement map, and a real subspace problem. Note that in many interesting situations one has a canonical relative orientation of the considered projections. In these situations one obtains canonical signs corresponding to the points of a regular fibre.

1. DEGREES OF REAL MAPS

By topological manifold we always mean a topological manifold which is Hausdorff and paracompact.

Let M be an n -dimensional topological manifold, and \mathcal{O}_M its orientation sheaf; for every point $m \in M$ we have

$$\mathcal{O}_{M,m} := H_n(M, M \setminus \{m\}, \mathbb{Z}) .$$

Note that for every locally constant sheaf ξ on M and for every open neighborhood U of m one has canonical identifications

$$\begin{aligned} H_n(M, M \setminus \{m\}, \xi) &= H_n(U, U \setminus \{m\}, \xi|_U) = H_n(U, U \setminus \{m\}, \xi_m) = \\ &= H_n(M, M \setminus \{m\}, \xi_m) = \mathcal{O}_{M,m} \otimes \xi_m , \end{aligned}$$

and

$$\begin{aligned} H^n(M, M \setminus \{m\}, \xi) &= H^n(U, U \setminus \{m\}, \xi|_U) = H^n(U, U \setminus \{m\}, \xi_m) = \\ &= H^n(M, M \setminus \{m\}, \xi_m) = \text{Hom}(\mathcal{O}_{M,m}, \xi_m) . \end{aligned}$$

Suppose now that M is connected and closed. The canonical fundamental class of M in cohomology is the canonical generator $\{M\}$ of $H^n(M, \mathcal{O}_M)$; for every $m \in M$ this class can be written as

$$\{M\} = j_m(c_m)$$

where $j_m : H^*(M, M \setminus \{m\}, \mathcal{O}_M) \rightarrow H^*(M, \mathcal{O}_M)$ is the natural morphism, and $c_m = \text{id}_{\mathcal{O}_{M,m}}$ is the canonical generator of $H^n(M, M \setminus \{m\}, \mathcal{O}_M)$.

Similarly, the canonical class of M in homology is the class $[M] \in H_n(M, \mathcal{O}_M)$ whose image in $H_n(M, M \setminus \{m\}, \mathcal{O}_M)$ is the canonical generator of this group for every $m \in M$.

Definition 1. Let M, N be differentiable manifolds, and $g : M \rightarrow N$ a continuous map. A relative orientation of g is an isomorphism $\nu : g^*(\mathcal{O}_N) \rightarrow \mathcal{O}_M$. A relatively oriented map is a pair (g, ν) , where ν is a relative orientation of g . A continuous map $g : M \rightarrow N$ is called relatively orientable if it admits a relative orientation.

Since w_1 classifies real line bundles on paracompact spaces, we obtain:

Remark 2. A continuous map $g : M \rightarrow N$ between differentiable manifolds is relatively orientable if and only if $g^*(w_1(T_N)) = w_1(T_M)$.

Remark 3. If $H^1(M, \mathbb{Z}_2) = 0$, then any continuous map $g : M \rightarrow N$ is relatively orientable. In particular any map $g : S^n \rightarrow N$ ($n \geq 2$) is relatively orientable.

Note that if $\dim(M) = \dim(N) =: n$ and (g, ν) is a relatively oriented differentiable map $M \rightarrow N$ between closed connected manifolds, one can define the degree $\deg_\nu(g)$ by the formula

$$(1) \quad \deg_\nu(g)\{M\} = \nu_*(g^*(\{N\})) .$$

If one just knows that g is relatively orientable, but did not fix a relative orientation of it, one can still define the *absolute degree* $|\deg|(g)$ of g by

$$|\deg|(g) := |\deg_\nu(g)|$$

for any relative orientation ν of g .

Let $x \in M$ be an isolated point in the fibre over its image $y := g(x)$, and choose charts $h : (U, x) \rightarrow (U', 0) \subset (\mathbb{R}^n, 0)$, $k : (V, y) \rightarrow (V', 0) \subset (\mathbb{R}^n, 0)$ such that the orientations of $T_x M$, $T_y N$ defined by the two charts correspond via ν_x . Then we define

$$\deg_{\nu,x}(g) := \deg_0(k \circ g \circ h^{-1}) ,$$

where the right hand term is computed with respect to the standard orientation of \mathbb{R}^n . This degree can be defined in an intrinsic way using the map

$$g_x : (U, U \setminus \{x\}) \rightarrow (N, N \setminus \{y\})$$

(which is well defined when U is sufficiently small) and writing

$$(2) \quad \nu_x(e_x g_x^*(c_y)) = \deg_{\nu,x}(g) c_x ,$$

where $e_x : H^n(U, U \setminus \{x\}, \mathcal{O}_y) \rightarrow H^n(M, M \setminus \{x\}, \mathcal{O}_y)$ is the isomorphism defined by excision.

Proposition 4. *Let M, N be closed connected n -manifolds, $g : M \rightarrow N$ a smooth map, $\nu : g^*(\mathcal{O}_N) \rightarrow \mathcal{O}_M$ a relative orientation, and $y \in N$ a point with $g^{-1}(y)$ finite. Then*

$$(3) \quad \deg_\nu(g) = \sum_{x \in g^{-1}(y)} \deg_{\nu,x}(g) .$$

Proof. Write $\{N\} = j_y(c_y)$, where $c_y \in H^n(N, N \setminus \{y\}, \mathcal{O}_N)$ is the canonical generator. One gets a pull-back class

$$g^*(c_y) \in H^n(M, M \setminus g^{-1}(y), g^*(\mathcal{O}_N)) ,$$

and $g^*(\{N\})$ can be written as

$$g^*(\{N\}) = J_{M,y}(g^*(c_y)) ,$$

where $J_{M,y} : H^n(M, M \setminus g^{-1}(y), g^*(\mathcal{O}_N)) \rightarrow H^n(M, g^*(\mathcal{O}_N))$ is the canonical map.

For every $x \in g^{-1}(y)$ let U_x be an open neighborhood of x such that $U_{x_1} \cap U_{x_2} = \emptyset$ for $x_1 \neq x_2$. Put $U := \bigcup_{x \in g^{-1}(y)} U_x$, denote by $g_x : (U_x, U_x \setminus \{x\}) \rightarrow (N, N \setminus \{y\})$ the restriction of g , and note that the inclusion $U \hookrightarrow M$ induces an isomorphism

$$H^n(U, U \setminus g^{-1}(y), (g|_U)^*(\mathcal{O}_N)) \rightarrow H^n(M, M \setminus g^{-1}(y), g^*(\mathcal{O}_N)) ,$$

by excision. We denote by

$$J_{U,y} : H^n(U, U \setminus g^{-1}(y), (g|_U)^*(\mathcal{O}_N)) \rightarrow H^n(M, g^*(\mathcal{O}_N))$$

the composition of $J_{M,y}$ with this isomorphism. One has an obvious isomorphism

$$H^n(U, U \setminus g^{-1}(y), (g|_U)^*(\mathcal{O}_N)) = \bigoplus_{x \in g^{-1}(y)} H^n(U_x, U_x \setminus \{x\}, \mathcal{O}_y)$$

and the restriction of $J_{U,y}$ to a summand $H^n(U_x, U_x \setminus \{x\}, \mathcal{O}_y)$ is the composition $j_x \circ e_x$ of the isomorphism $e_x : H^n(U_x, U_x \setminus \{x\}, \mathcal{O}_y) \rightarrow H^n(M, M \setminus \{x\}, \mathcal{O}_y)$ with the canonical map $J_x : H^*(M, M \setminus \{x\}, \mathcal{O}_y) \rightarrow H^*(M, g^*(\mathcal{O}_N))$. Therefore

$$\begin{aligned} \nu g^*(\{N\}) &= \nu J_{U,y}((g|_U)^*(c_y)) = \nu J_{U,y} \left(\sum_{x \in g^{-1}(y)} g_x^*(c_y) \right) = \nu \left(\sum_{x \in g^{-1}(y)} J_x \circ e_x(g_x^*(c_y)) \right) \\ &= \sum_{x \in g^{-1}(y)} \nu_x \circ J_x \circ e_x(g_x^*(c_y)) = \sum_{x \in g^{-1}(y)} j_x \circ \nu_x \circ e_x(g_x^*(c_y)) = \sum_{x \in g^{-1}(y)} \deg_{\nu,x}(g) j_x(c_x) \\ &= \left\{ \sum_{x \in g^{-1}(y)} \deg_{\nu,x}(g) \right\} \{M\} . \end{aligned}$$

by (2). ■

Proposition 5. *Let X, Y be compact complex manifolds endowed with real structures, let $f : X \rightarrow Y$ be a finite Real holomorphic map such that the induced map $f(\mathbb{R}) : X(\mathbb{R}) \rightarrow Y(\mathbb{R})$ is relatively oriented. Then*

(1) For any $y \in Y(\mathbb{R})$ one has

$$|\deg|(f(\mathbb{R})) \leq \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) \leq \deg(f) .$$

(2) $\deg(f(\mathbb{R})) \equiv \deg(f) \pmod{2}$.

(3) For any $y \in Y(\mathbb{R})$ one has

$$\deg(f(\mathbb{R})) \equiv \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) \pmod{2} .$$

Proof. Choose a relative orientation $\nu : f(\mathbb{R})^*(\mathcal{O}_{Y(\mathbb{R})}) \rightarrow \mathcal{O}_{X(\mathbb{R})}$.

(1) For the first inequality note that

$$\begin{aligned} |\deg|(f(\mathbb{R})) &= \left| \sum_{x \in f(\mathbb{R})^{-1}(y)} \deg_{\nu, x}(f(\mathbb{R})) \right| \leq \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) = \\ &= \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f . \end{aligned}$$

For the second inequality, note that any point $y \in Y(\mathbb{R})$, we have

$$\begin{aligned} \deg(f) &= \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f + \sum_{x \in f^{-1}(y) \setminus X(\mathbb{R})} \text{mult}_x f = \\ (4) \quad &= \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) + \sum_{x \in f^{-1}(y) \setminus X(\mathbb{R})} \text{mult}_x f \end{aligned}$$

and all the terms on the right are positive.

(2) Choose a regular value $y \in Y(\mathbb{R})$ of $f(\mathbb{R})$. Then one has $\deg_{\nu, x}(f(\mathbb{R})) \in \{\pm 1\}$ for every $x \in f^{-1}(y)$ and

$$\deg(f(\mathbb{R})) = \sum_{x \in f(\mathbb{R})^{-1}(y)} \deg_{\nu, x}(f(\mathbb{R})) \equiv \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) \pmod{2} .$$

It suffices to note that

$$\text{mult}_x f(\mathbb{R}) = \text{mult}_x f , \text{ and } \sum_{x \in f^{-1}(y) \setminus X(\mathbb{R})} \text{mult}_x f \equiv 0 \pmod{2} .$$

(3) We use (2) and note that, by (1) one obviously has

$$\deg(f) \equiv \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) \pmod{2} .$$

■

Note that the estimate (1) holds without any transversality assumption on f . A statement similar to Proposition 5 holds for Real sections in Real holomorphic vector bundles over compact complex manifolds endowed with Real structures [OT2].

Proposition 6. *Let X be compact complex manifold endowed with real structure, and let $E \rightarrow X$ be a Real holomorphic vector bundle over X with $\text{rk}(E) = \dim(X) = n$. Suppose that the real vector bundle $E(\mathbb{R}) \rightarrow X(\mathbb{R})$ is relatively orientable, and let s be a Real holomorphic section of E with finite zero locus $Z(s)$. Then*

(1)

$$|\deg|(E(\mathbb{R})) \leq \sum_{z \in Z(s) \cap X(\mathbb{R})} \text{mult}_z(s) \leq \langle c_n(E), [X] \rangle .$$

(2)

$$|\deg|(E(\mathbb{R})) \equiv \langle c_n(E), [X] \rangle \pmod{2} .$$

(3)

$$|\deg|(E(\mathbb{R})) \equiv \sum_{z \in Z(s) \cap X(\mathbb{R})} \text{mult}_z(s) .$$

2. WALL CROSSING FOR DEGREES OF REAL PROJECTIONS

2.1. Projections. The wall associated with a submanifold of $\mathbb{P}(V)$. Let V be an N -dimensional real vector space and let λ_V be the tautological line bundle on the projective space $\mathbb{P}(V)$. By definition, λ_V is a line subbundle of the trivial bundle $\underline{V} := \mathbb{P}(V) \times V$ and the tangent bundle $T_{\mathbb{P}(V)}$ can be canonically identified with $\text{Hom}(\lambda_V, \underline{V}/\lambda_V) = \lambda_V^\vee \otimes \underline{V}/\lambda_V$.

Let $X \subset \mathbb{P}(V)$ be a compact submanifold of dimension $m < N - 1$. We denote by T_X the tangent bundle of X regarded as a subbundle of the restriction $T_{\mathbb{P}(V)}|_X$, and by $\underline{V}_X, \lambda_{V,X}$ the restrictions of the bundles \underline{V}, λ_V to X . The tensor product $T_X \otimes \lambda_{V,X}$ is a subbundle of the quotient bundle $\underline{V}_X/\lambda_{V,X}$, so it can be written as $Y_X/\lambda_{V,X}$, where $Y_X \subset \underline{V}_X$ is the subbundle of \underline{V}_X defined as the preimage of $T_X \otimes \lambda_{V,X}$ under the epimorphism

$$\underline{V}_X \twoheadrightarrow \underline{V}_X/\lambda_{V,X} .$$

With this definition we obtain a canonical identification

$$(5) \quad T_X = \text{Hom}(\lambda_{V,X}, Y_X/\lambda_{V,X}) ,$$

which will play an important role in the following constructions. Note that the bundle Y_X can be identified with the dual jet bundle $[J^1(\lambda_{V,X}^\vee)]^\vee$ of $\lambda_{V,X}^\vee$.

Let now W be a real vector space of dimension $m + 1$. A morphism $f \in \text{Hom}(V, W)$ defines the central projection

$$[f] : \mathbb{P}(V) \setminus \mathbb{P}(\ker(f)) \rightarrow \mathbb{P}(W) ,$$

whose restriction to $X \setminus \mathbb{P}(\ker(f))$ will be denoted by $[f]_X$.

The differential of a central projection $[f]$ can be computed as follows: For a line $l \in \mathbb{P}(V) \setminus \mathbb{P}(\ker(f))$ we have an induced isomorphism

$$r_{f,l} : l \rightarrow f(l) ,$$

and an induced linear map

$$q_{f,l} : V/l \rightarrow W/f(l) .$$

Let H' be a linear complement of the line $f(l)$ in W , and note that $H := f^{-1}(H')$ is a linear complement of l in V . For $\varphi \in \text{Hom}(l, H)$, the $[f]$ -image of the line $l_\varphi := \{v + \varphi(v) \mid v \in l\}$ in $\mathbb{P}(W)$ is $\{f(v) + f(\varphi(v)) \mid v \in l\}$, which is the graph of $f \circ \varphi \circ r_{f,l}^{-1}$. This shows that via the identifications

$$T_l(\mathbb{P}(V)) = \text{Hom}(l, V/l) , \quad T_{f(l)}(\mathbb{P}(W)) = \text{Hom}(f(l), W/f(l))$$

the differential $[f]_{*l}$ at a point $l \in \mathbb{P}(V) \setminus \mathbb{P}(\ker(f))$ is given by

$$(6) \quad [f]_{*,l}(\varphi) = q_{f,l} \circ \varphi \circ r_{f,l}^{-1}.$$

The differential of the restriction $[f]_X$ is given by the same formula applied to $\varphi \in T_l(X) = \text{Hom}(l, Y_l/l)$. More precisely:

Remark 7. *The differential of the restriction $[f]_X$ at $l \in X \setminus \mathbb{P}(\ker(f))$ is given by the formula:*

$$(7) \quad ([f]_X)_{*,l}(\varphi) = p_{f,l} \circ \varphi \circ r_{f,l}^{-1}.$$

where $p_{f,l} : Y_l/l \rightarrow W/f(l)$ is the isomorphism induced by f .

Remark 8. *Let $l \in X \setminus \mathbb{P}(\ker(f))$. The following conditions are equivalent:*

- (1) $[f]_X$ is a local diffeomorphism at l ,
- (2) $f^{-1}(f(l)) \cap Y_l = l$,
- (3) $\ker(f) \cap Y_l = \{0\}$.

Proof. Using Remark 7 we see that $([f]_X)_{*,l}$ is an isomorphism if and only if the restriction of

$$q_{f,l} : V/l \rightarrow W/f(l)$$

to Y_l/l is an isomorphism. Since the kernel of this restriction is $[f^{-1}(f(l)) \cap Y_l]/l$, we obtain the equivalence (1) \Leftrightarrow (2). On the other hand $f^{-1}(f(l)) = l + \ker(f)$, hence $f^{-1}(f(l)) \cap Y_l = (l + \ker(f)) \cap Y_l$. The composition

$$\ker(f) \cap Y_l \hookrightarrow (l + \ker(f)) \cap Y_l \rightarrow (l + \ker(f)) \cap Y_l/l$$

is surjective because $l \subset Y_l$, and it is injective since $l \cap \ker(f) = \{0\}$. Therefore one has $f^{-1}(f(l)) \cap Y_l = l$ if and only if $\ker(f) \cap Y_l = \{0\}$. ■

In the special case $X \cap \mathbb{P}(\ker(f)) = \emptyset$ we obtain a well-defined smooth map $[f]_X : X \rightarrow \mathbb{P}(W)$ between compact manifolds of the same dimension. The main goal of this section is to study the relative orientability of such a map and, in the relatively orientable case, to study the possible degrees of $[f]_X$ for a fixed submanifold X .

Since the projection $[f]_X$ is defined on all of X if and only if f belongs to the open subset

$$\text{Hom}(V, W)_X := \{f \in \text{Hom}(V, W) \mid X \cap \mathbb{P}(\ker(f)) = \emptyset\},$$

it is natural to study the complement of this open set, namely the *wall* associated with X .

Definition 9. *Let $X \subset \mathbb{P}(V)$ be a compact m -dimensional submanifold and let W be a real vector space of dimension $n := m + 1$. The wall associated with X is defined by*

$$\mathcal{W}_X := \{f \in \text{Hom}(V, W) \mid X \cap \mathbb{P}(\ker(f)) \neq \emptyset\}.$$

A point $f \in \mathcal{W}_X$ is called *regular* if the following conditions are satisfied:

- (1) $\dim(\ker(f)) = N - n$ or, equivalently, f is an epimorphism,
- (2) The intersection $X \cap \mathbb{P}(\ker(f))$ has only one point, denoted by ξ_f ,
- (3) $\ker(f) \cap Y_{\xi_f} = \xi_f$.

Denote by \mathcal{W}_X^0 the subspace of regular points of the wall.

The following proposition shows that the wall $\mathcal{W}_X \subset \text{Hom}(V, W)$ is a smooth hypersurface at any regular point f_0 and identifies the normal line of this hypersurface at f_0 canonically with a line depending only on the triple (X, W, ξ_{f_0}) .

Proposition 10. *Let $X \subset \mathbb{P}(V)$ be a smooth m -dimensional submanifold. Then:*

- (1) *The wall $\mathcal{W}_X \subset \text{Hom}(V, W)$ is a smooth hypersurface at any regular point $f_0 \in \mathcal{W}_X$.*
- (2) *Denoting $X \cap \mathbb{P}(\ker(f_0)) = \{\xi_0\}$ we have a canonical isomorphism*

$$\psi_{f_0} : N_{\mathcal{W}_X, f_0} \rightarrow \text{Hom}(\det(Y_{\xi_0}), \det(W)) .$$

Proof. 1. Consider the incidence varieties

$$\mathcal{J} := \{(f, \xi, K) \in \text{Hom}(V, W) \times \mathbb{P}(V) \times G_{N-n}(V) \mid \xi \subset K \subset \ker(f)\} .$$

$$\mathcal{J}_X := \{(f, \xi, K) \in \text{Hom}(V, W) \times X \times G_{N-n}(V) \mid \xi \subset K \subset \ker(f)\} .$$

It's easy to see that \mathcal{J} (respectively \mathcal{J}_X) is a vector bundle over submanifolds of $\mathbb{P}(V) \times G_{N-n}(V)$ (respectively $X \times G_{N-n}(V)$), so it has a natural manifold structure. Let $q : \mathcal{J} \rightarrow \text{Hom}(V, W)$, $q_X : \mathcal{J}_X \rightarrow \text{Hom}(V, W)$ be the projections on the first factor, which are obviously proper smooth maps.

Note that

$$(8) \quad \mathcal{W}_X = q_X(\mathcal{J}_X) .$$

One has a canonical identification

$$T_{(f, \xi, K)}(\mathcal{J}) = \{(\varphi, \alpha, \beta) \in \text{Hom}(V, W) \times \text{Hom}(\xi, V/\xi) \times \text{Hom}(K, V/K) \mid \varphi|_K = f \circ \beta, \beta|_\xi = \bar{\alpha}\} ,$$

where $\bar{\alpha}$ denotes the composition of α with the natural epimorphism $V/\xi \rightarrow V/K$. Via this identification we have $q_{*(f, \xi, K)}(\varphi, \alpha, \beta) = \varphi$, hence

$$\ker(q_{*(f, \xi, K)}) = \{(\alpha, \beta) \in \text{Hom}(\xi, V/\xi) \times \text{Hom}(K, \ker(f)/K) \mid \beta|_\xi = \bar{\alpha}\} .$$

Similarly

$$\ker((q_X)_{*(f, \xi, K)}) = \{(\alpha, \beta) \in \text{Hom}(\xi, Y_\xi/\xi) \times \text{Hom}(K, \ker(f)/K) \mid \beta|_\xi = \bar{\alpha}\} .$$

Let now $f_0 \in \mathcal{W}_X^0$ be a regular point on the wall, and put $K_0 := \ker(f_0)$, $\xi_0 := \xi_{f_0}$. We will show that q_X is an immersion at (f_0, ξ_0, K_0) . Indeed, since we have $K_0 = \ker(f_0)$, the kernel $\ker((q_X)_{*(f_0, \xi_0, K_0)})$ can be identified with

$$\{\alpha \in \text{Hom}(\xi_0, Y_{\xi_0}/\xi_0) \mid \bar{\alpha} = 0\} = \text{Hom}(\xi_0, (K_0 \cap Y_{\xi_0})/\xi_0) ,$$

which vanishes because $K_0 \cap Y_{\xi_0} = \xi_0$. Therefore q_X is an immersion at (f_0, ξ_0, K_0) as claimed. Since the fibre $q_X^{-1}(f_0)$ has only one element and q_X is proper, it follows that $q_X(U)$ is a neighborhood of f_0 for every neighborhood U of (f_0, ξ_0, K_0) in \mathcal{J}_X . This implies that the image $\mathcal{W}_X = q_X(\mathcal{J}_X)$ is a submanifold of $\text{Hom}(V, W)$ at f_0 whose germ at f_0 can be identified with the germ at (f_0, ξ_0, K_0) of \mathcal{J}_X . A simple dimension count shows that \mathcal{W}_X is a hypersurface at f_0 .

2. For the second statement of the proposition we will construct a canonical isomorphism

$$\psi_{f_0} : N_{\mathcal{W}_X, f_0} \rightarrow \text{Hom}(\det(Y_{\xi_0}), \det(W))$$

as the composition $u_{f_0} \circ a_{f_0}$ of two canonical isomorphisms:

$$a_{f_0} : N_{\mathcal{W}_X, f_0} \rightarrow \text{Hom}\left(\xi_0, W/f_0(Y_{\xi_0})\right) ,$$

$$u_{f_0} : \text{Hom}\left(\xi_0, W/f_0(Y_{\xi_0})\right) \rightarrow \text{Hom}(\det(Y_{\xi_0}), \det(W)) .$$

We define

$$A_{f_0} : \text{Hom}(V, W) = T_{f_0} \text{Hom}(V, W) \rightarrow \text{Hom}\left(\xi_0, W/f_0(Y_{\xi_0})\right)$$

by

$$(9) \quad A_{f_0}(\varphi) := \varphi|_{\xi_0} \bmod f_0(Y_{\xi_0}) .$$

It is easy to see that $A_{f_0}(\varphi) = 0$ if and only if there exists

$$(\alpha, \beta) \in \text{Hom}(\xi_0, Y_{\xi_0}/\xi_0) \times \text{Hom}(K_0, V/K_0)$$

such that $(\varphi, \alpha, \beta) \in T_{(f_0, \xi_0, K_0)}(\mathcal{J}_X)$, i.e., if and only if $\varphi \in T_{f_0}(\mathcal{W}_X^0)$. Hence A_{f_0} induces a canonical isomorphism $a_{f_0} : N_{\mathcal{W}_X^0, f_0} \rightarrow \text{Hom}(\xi_0, W/f_0(Y_{\xi_0}))$ as claimed.

For the construction of u_{f_0} we use the exact sequences:

$$0 \rightarrow \xi_0 \hookrightarrow Y_{\xi_0} \rightarrow Y_{\xi_0}/\xi_0 \rightarrow 0 ,$$

$$0 \rightarrow f_0(Y_{\xi_0}) \hookrightarrow W \rightarrow W/f_0(Y_{\xi_0}) \rightarrow 0 ,$$

which give standard isomorphisms

$$(10) \quad \det(Y_{\xi_0}) = \xi_0 \otimes \det(Y_{\xi_0}/\xi_0) ,$$

$$(11) \quad \det(W) = \det(f_0(Y_{\xi_0})) \otimes W/f_0(Y_{\xi_0})$$

Note that one has a second canonical (but non-standard!) isomorphism

$$(12) \quad \det(W) = W/f_0(Y_{\xi_0}) \otimes \det(f_0(Y_{\xi_0}))$$

defined by $[w] \otimes \delta \mapsto w \wedge \delta$ (see Remark 11 below), which is more convenient in our situation, because the maps we consider here relate the factors of the tensor products in (10), (12) respecting the order.

The isomorphism $u_{f_0} : \text{Hom}\left(\xi_0, W/f_0(Y_{\xi_0})\right) \rightarrow \text{Hom}(\det(Y_{\xi_0}), \det(W))$ is defined via the isomorphisms (10) and (12) by

$$(13) \quad u_{f_0}(\sigma) = \sigma \otimes \det(\bar{f}_0) ,$$

where $\bar{f}_0 : Y_{\xi_0}/\xi_0 \rightarrow f_0(Y_{\xi_0})$ is the isomorphism induced by f_0 .

Note that if one uses the standard isomorphism (11) for $\det(W)$ the corresponding formula for u_{f_0} would be

$$u_{f_0}(\sigma)(v \otimes \delta) := (-1)^m(\sigma(v) \otimes [\det(\bar{f}_0)(\delta)]) .$$

■

Remark 11. Let C be finite dimensional real vector space. For every subspace $A \subset C$ we define the isomorphisms

$$u_A : \det(A) \otimes \det(C/A) \xrightarrow{\cong} \det(C), \quad v_A : \det(C/A) \otimes \det(A) \xrightarrow{\cong} \det(C)$$

by

$$u_A(\delta \otimes [c]) = \delta \wedge c, \quad v_A([c] \otimes \delta) = c \wedge \delta .$$

(1) Let

$$\mathbf{p} : \det(A) \otimes \det(C/A) \rightarrow \det(C/A) \otimes A$$

be the obvious isomorphism defined by permutation of the factors, and put $a := \dim(A)$, $c := \dim(C)$. Then one has

$$v_A^{-1} \circ u_A = (-1)^{a(c-a)} \mathbf{p} .$$

(2) Suppose C has an internal direct sum decomposition $C = A \oplus B$, and let $\alpha : A \xrightarrow{\cong} C/B$, $\beta : B \xrightarrow{\cong} C/A$ be the obvious isomorphisms. Then one has

$$\det(\alpha) \otimes \det(\beta)^{-1} = v_B^{-1} \circ u_A .$$

In other words, via the isomorphisms u_A , v_B the tensor product $\det(\alpha) \otimes \det(\beta)^{-1}$ induces $\text{id}_{\det(C)}$.

2.2. Relative orientations of central projections. Our next goal is to describe explicitly the set of relative orientations of a map $[f]_X$ associated with a morphism $f \in \text{Hom}(V, W)_X$. We will obtain an important (and surprising) result: relative orientability of a map $[f]_X$ depends only on the embedding $X \subset \mathbb{P}(V)$, and the set of relative orientations of $[f]_X$ can be canonically identified with a set which is intrinsically associated with this embedding, and is independent of the choice of $f \in \text{Hom}(V, W)$. First we give a simple description of the line bundle $[f]^*(\det(T_{\mathbb{P}(W)}))$ on $\mathbb{P}(V) \setminus \mathbb{P}(\ker(f))$:

Let again λ_V (respectively λ_W) be the tautological line bundle on $\mathbb{P}(V)$ (respectively $\mathbb{P}(W)$). The family of isomorphisms

$$(r_{f,l} : l \rightarrow f(l))_{l \in \mathbb{P}(V) \setminus \mathbb{P}(\ker(f))}$$

defines a line bundle isomorphism

$$r_f : \lambda_V|_{\mathbb{P}(V) \setminus \mathbb{P}(\ker(f))} \xrightarrow{\cong} [f]^*(\lambda_W) .$$

Using the canonical isomorphism

$$\det(T_{\mathbb{P}(W)}) = [\lambda_W^\vee]^{\otimes n} \otimes \det(W) ,$$

we obtain a canonical isomorphism

$$(14) \quad [f]^*(\det(T_{\mathbb{P}(W)})) \xrightarrow{(r_f^\vee)^{\otimes n} \otimes \text{id}} [\lambda_V^\vee|_{\mathbb{P}(V) \setminus \mathbb{P}(\ker(f))}]^{\otimes n} \otimes \det(W) .$$

We can prove now

Lemma 12. *Let $f \in \text{Hom}(V, W)_X$ and let $[f]_X : X \rightarrow \mathbb{P}(W)$ be the projection induced by f . Then:*

(1) *There is a canonical isomorphism*

$$[f]_X^*(\det(T_{\mathbb{P}(W)})) = [\lambda_{V,X}^\vee]^{\otimes n} \otimes \det(W) .$$

(2) *The isomorphism class of the line bundle $[f]_X^*(\det(T_{\mathbb{P}(W)}))$ on X is independent of $f \in \text{Hom}(V, W)_X$.*

(3) *The restriction $[f]_X : X \rightarrow \mathbb{P}(W)$ is relatively orientable if and only if $w_1(X) = n\{w_1(\lambda_V)|_X\}$.*

(4) *The data of an isomorphism*

$$\nu : [f]_X^*(\det(T_{\mathbb{P}(W)})) \rightarrow \det(T_X)$$

is equivalent to the data of a line bundle isomorphism

$$\mu : X \times \det(W) \xrightarrow{\cong} \det(Y_X),$$

hence to the data of a global trivialization of $\det(Y_X)$ with fibre $\det(W)$.

Proof. The first statement follows by restricting the isomorphism (14) to X . The statements (2) and (3) are direct consequences of (1) whereas (4) follows from (1) and the canonical isomorphism

$$\det(T_X) = [\lambda_{V,X}^\vee]^{\otimes n} \otimes \det(Y_X)$$

induced by (5). ■

This proves the following important

Proposition 13. *Relative orientability of a map $[f]_X : X \rightarrow \mathbb{P}(W)$ defined by $f \in \text{Hom}(V, W)_X$ depends only on the embedding $X \subset \mathbb{P}(V)$, and is independent of f . The set of relative orientations of such a map $[f]_X : X \rightarrow \mathbb{P}(W)$ can be canonically identified with the set of connected components of the space of line bundle isomorphisms*

$$\mu : X \times \det(W) \xrightarrow{\cong} \det(Y_X),$$

hence this set is independent of f .

Definition 14. *A bundle isomorphism $\mu : X \times \det(W) \xrightarrow{\cong} \det(Y_X)$ will be called orientation parameter for projections $X \rightarrow \mathbb{P}(W)$. Given $f \in \text{Hom}(V, W)_X$ we denote by*

$$\nu(f, \mu) := [(r_f^\vee)^{\otimes n} \otimes \mu]$$

the relative orientation of $[f]_X$ associated with orientation parameter μ .

Therefore for every orientation parameter $\mu : X \times \det(W) \xrightarrow{\cong} \det(Y_X)$ we obtain a well defined map

$$\deg_\mu^X : \text{Hom}(V, W)_X = \text{Hom}(V, W) \setminus \mathcal{W}_X \rightarrow \mathbb{Z},$$

which obviously descends to a map (denoted by the same symbol):

$$\deg_\mu^X : \pi_0(\text{Hom}(V, W) \setminus \mathcal{W}_X) \rightarrow \mathbb{Z}$$

Inspired by the terminology used in gauge theory we introduce the following

Definition 15. *The connected components of $\text{Hom}(V, W) \setminus \mathcal{W}_X$ will be called chambers.*

In the next section we will focus on the following problem: how can one compare the values of the degree map on different chambers. The first step will be a wall-crossing formula which computes the jump of the degree map when one crosses the wall transversally at a regular point. The main ingredient in proving this wall-crossing formula is the following remark which states that, for a given point $\xi_0 \in X$, the set of isomorphism $\det(Y_{\xi_0}) \rightarrow \det(W)$ has two (radically different) geometric interpretations:

Remark 16. *Fix $\xi_0 \in X$. The orientation parameter μ defines an isomorphism $\mu_{\xi_0}^{-1} : \det(Y_{\xi_0}) \rightarrow \det(W)$, which (by Proposition 10) also defines a generator of the normal line $N_{\mathcal{W}_X^0, f}$ for every $f \in \mathcal{W}_X^0$ for which $\xi_f = \xi_0$.*

This important remark will play a crucial role in the following section.

Fix now an orientation parameter μ , and suppose that $[f]_X$ is well-defined and a local isomorphism at $\xi_0 \in X$. By Remark 8 this means that $\ker(f) \cap Y_{\xi_0} = \{0\}$. We need an explicit formula for the local degree $\deg_{\nu(f,\mu),\xi_0}([f]_X)$. Using the exact sequences

$$0 \rightarrow \xi_0 \rightarrow Y_{\xi_0} \rightarrow Y_{\xi_0}/\xi_0 \rightarrow 0, \quad 0 \rightarrow f(\xi_0) \rightarrow W \rightarrow W/f(\xi_0) \rightarrow 0,$$

we get canonical isomorphisms

$$\begin{aligned} \det(Y_{\xi_0}) &= \xi_0 \otimes \det(Y_{\xi_0}/\xi_0), \\ (15) \quad \det(W) &= f(\xi_0) \otimes \det(W/f(\xi_0)). \end{aligned}$$

By Remark 7 the differential

$$[f]_{*,\xi_0} : T_{\xi_0}(X) = \text{Hom}(\xi_0, Y_{\xi_0}/\xi_0) \rightarrow \text{Hom}(f(\xi_0), W/f(\xi_0)) = T_{f(\xi_0)}\mathbb{P}(W)$$

is given by

$$[f]_{*,\xi_0}(\varphi) = p_{f,\xi_0} \circ \varphi \circ r_{f,\xi_0}^{-1},$$

where $r_{f,\xi_0} : \xi_0 \rightarrow f(\xi_0)$, $p_{f,\xi_0} : Y_{\xi_0}/\xi_0 \rightarrow W/f(\xi_0)$ are the obvious isomorphisms induced by f . Therefore

Lemma 17. *Let μ be an orientation parameter, and suppose that $[f]_X$ is well-defined and a local isomorphism at $\xi_0 \in X$. Via the isomorphisms (2.2) the local degree $\deg_{\nu(f,\mu),\xi_0}([f]_X)$ is given by*

$$(16) \quad \deg_{\nu(f,\mu),\xi_0}([f]_X) = \text{sign}(\mu_{\xi_0} \circ (r_{f,\xi_0} \otimes \det(p_{f,\xi_0}))) .$$

We will also need the following simple remark concerning the functoriality of the degree map with respect to isomorphisms $\Phi : W \rightarrow W'$.

Remark 18. *Let $\mu : X \times \det(W) \rightarrow \det(Y_X)$ be an orientation parameter, $f \in \text{Hom}(V, W)_X$ and $\Phi : W \rightarrow W'$ a vector space isomorphism. Then*

$$\deg_{\nu(f,\mu \circ \det(\Phi)^{-1})}([\Phi \circ f]_X) = \deg_{\nu(f,\mu)}^X([f]_X) .$$

In particular, if $W = W'$ one has:

$$\deg_{\nu(f,\mu)}^X([\Phi \circ f]_X) = \text{sign}(\det(\Phi)) \deg_{\nu(f,\mu)}^X([f]_X) .$$

We end this section with an example.

Example: (Veronese maps) Let W be a real vector space of dimension $n \geq 2$, and let $g : \mathbb{P}(W) \rightarrow \mathbb{P}(W)$ be a regular real algebraic map. Such a map factorizes as $g = [f] \circ v_d$, where

$$v_d : \mathbb{P}(W) \rightarrow \mathbb{P}(S^d W)$$

is the Veronese map of degree d , and $f \in \text{Hom}(S^d(W), W)$ with $\mathbb{P}(\ker(f)) \cap \text{im}(v_d) = \emptyset$. The positive integer d is determined by g and will be called the algebraic degree of g . Applying Lemma 12 to the image $X := v_d(\mathbb{P}(W))$ and noting that $v_d^*(\lambda_{S^d(W)}) = \lambda_W^{\otimes d}$, one obtains:

$$\det(T_{\mathbb{P}(W)})^\vee \otimes g^*(\det(T_{\mathbb{P}(W)})) = \lambda_W^{\otimes [n(1-d)]}$$

This proves the following simple, but interesting, result:

Proposition 19. *A regular real algebraic map $g : \mathbb{P}(W) \rightarrow \mathbb{P}(W)$ of algebraic degree d is relatively orientable if and only if $\dim(W)(1-d)$ is even, and in this case it is canonically relatively oriented.*

In the case when $\dim(W)(1-d)$ is even, it is an interesting problem to determine the possible degrees of such regular real algebraic maps with respect to this canonical relative orientation. The case $\dim(W) = 2$ is known by the work of Brockett (see [Se] p. 41 or [By] Theorem 2.1) and will be described in detail at the end of section 2.3.

2.3. Wall crossing jump. Suppose now that the condition in the third statement of Lemma 12 is satisfied, and fix an isomorphism

$$\mu : X \times \det(W) \rightarrow \det(Y_X).$$

We are interested in the map $\deg_\mu^X : \pi_0(\text{Hom}(V, W)_X) \rightarrow \mathbb{Z}$ defined by

$$\deg_\mu^X(\hat{f}) := \deg_{\nu(f, \mu)}([f]_X).$$

Note that $[f]$ can be written as the composition

$$X \hookrightarrow \mathbb{P}(V) \setminus \mathbb{P}(\ker(f)) \rightarrow \mathbb{P}(\text{im}(f)) \hookrightarrow \mathbb{P}(W),$$

where the central map is the projection of $\mathbb{P}(W)$ onto $\mathbb{P}(\text{im}(f))$ with center $\mathbb{P}(\ker(f))$. In other words, we are interested in the variation of the degree of central projections $\mathbb{P}(V) \supset X \rightarrow \mathbb{P}(W)$ when the projection varies. The first step is to determine the variation of $\deg_{\nu(f, \mu)}([f]_X)$ as f varies on a curve which crosses the wall transversally at a regular point.

Theorem 20. (*Wall-crossing formula*) *Let $f_0 \in \mathcal{W}_X^0$ be a regular point on the wall, put $\xi_0 := \xi_{f_0}$, $N_0 := N_{\mathcal{W}_X^0, f_0}$, and let*

$$\mathfrak{f} = (f_\tau)_{\tau \in N_0} : N_0 \rightarrow \text{Hom}(V, W)$$

be a smooth map such that the differential $\mathfrak{f}_{,0}$ is a right splitting of the exact sequence*

$$0 \rightarrow T_{\mathcal{W}_X^0, f_0} \rightarrow T_{\text{Hom}(V, W), f_0} = \text{Hom}(V, W) \rightarrow N_0 \rightarrow 0.$$

Then for every sufficiently small $\tau \in N_0 \setminus \{0\}$ we have:

- (1) $f_\tau \in \text{Hom}(V, W)_X$ and $[f_\tau]_X$ is a local diffeomorphism at ξ_0 ,
- (2) $\deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X) = \text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\tau)))$,
- (3) $\deg_{\nu(f_\tau, \mu)}([f_\tau]_X) - \deg_{\nu(f_{-\tau}, \mu)}([f_{-\tau}]_X) = 2\text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\tau)))$.

Proof. (1) The fact that $f_\tau \in \text{Hom}(V, W)_X$ for sufficiently small $\tau \in N_0 \setminus \{0\}$ follows directly from Proposition 10 taking into account that the complement $\mathcal{W}_X = \text{Hom}(V, W) \setminus \text{Hom}(V, W)_X$ is a smooth hypersurface at f_0 and the curve $(f_\tau)_{\tau \in N_0}$ is transversal to this hypersurface at f_0 . In order to prove that $[f_\tau]_X$ is a local diffeomorphism at ξ_0 we use Remark 8. We have to show that $\ker(f_\tau) \cap Y_{\xi_0} = \{0\}$.

Note that, in general, for two finite dimensional real vector spaces A, B of the same dimension the closed subset

$$\mathcal{W}(A, B) := \{s \in \text{Hom}(A, B) \mid \ker(s) \neq \{0\}\}$$

of $\text{Hom}(A, B)$ is a smooth hypersurface at any point s_0 with $\dim(\ker(s_0)) = 1$, and the tangent space at such a point is

$$T_{s_0} \mathcal{W}(A, B) = \{\sigma \in \text{Hom}(A, B) \mid \sigma(\ker(s_0)) \subset s_0(A)\}.$$

Therefore a tangent vector $\sigma \in T_{s_0}(\text{Hom}(A, B))$ is transversal to $\mathcal{W}(A, B)$ at s_0 if and only if the linear map $\ker(s_0) \rightarrow B/s_0(A)$ induced by $\sigma|_{\ker(s_0)}$ is an isomorphism.

Using this remark we see that the map $\tilde{f} = (f_\tau)_\tau : N_0 \rightarrow \text{Hom}(Y_{\xi_0}, W)$ given by $\tilde{f}_\tau = f_\tau|_{Y_{\xi_0}}$ is transversal to $\mathcal{W}(Y_{\xi_0}, W)$ at f_0 . Indeed, using the notations introduced in the proof of Proposition 10 the map

$$\ker \tilde{f}_0 = \xi_0 \longrightarrow W/f_0(Y_{\xi_0})$$

induced by $\tilde{f}_{*,0}(\tau)$ is precisely $A_{f_0}(\tilde{f}_{*,0}(\tau))$ by definition of A_{f_0} . On the other hand, since $\tilde{f}_{*,0}$ is a right inverse of the canonical projection $\text{Hom}(V, W) \rightarrow N_0$, we see that $A_{f_0}(\tilde{f}_{*,0}(\tau)) = a_{f_0}(\tau)$, which is nonzero for $\tau \in N_0 \setminus \{0\}$ because a_{f_0} is an isomorphism by Proposition 10.

(2) We suppose first that f is an affine map, so it has the form

$$(17) \quad f_\tau = f_0 + \phi(\tau) ,$$

where $\phi : N_0 \rightarrow \text{Hom}(V, W)$ is a linear map (which coincides with the differential $\tilde{f}_{*,0}$). As we have seen in the proof of (1) our assumption about the differential $\tilde{f}_{*,0}$ implies that $\phi(\tau)$ is a lift of τ , so that $a_{f_0}(\tau) = A_{f_0}(\phi(\tau))$. Therefore for $\tau \neq 0$ the morphism

$$a_{f_0}(\tau) = A_{f_0}(\phi(\tau)) = \bar{\phi}(\tau)|_{\xi_0} : \xi_0 \rightarrow W/f_0(Y_{\xi_0})$$

induced by $\phi(\tau)$ is an isomorphism. For every $\tau \in N_0 \setminus \{0\}$ we obtain a direct sum decomposition

$$(18) \quad W = \phi(\tau)(\xi_0) \oplus f_0(Y_{\xi_0}) ,$$

which is independent of τ , since N_0 is 1-dimensional. Now fix $\tau \in N_0 \setminus \{0\}$. We have an obvious commutative diagram with exact rows

$$(19) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \xi_0 & \longrightarrow & Y_{\xi_0} & \longrightarrow & Y_{\xi_0}/\xi_0 & \longrightarrow & 0 \\ & & \downarrow r_{\phi(\tau), \xi_0} & & \downarrow f_\tau|_{Y_{\xi_0}} & & \downarrow p_0 + \phi(\tau)_0 & & \\ 0 & \longrightarrow & \xi_0 & \longrightarrow & W & \longrightarrow & W/\phi(\tau)(\xi_0) & \longrightarrow & 0 \end{array} ,$$

where

$$p_0 : Y_{\xi_0}/\xi_0 \rightarrow W/\phi(\tau)(\xi_0) , \quad \phi(\tau)_0 : Y_{\xi_0}/\xi_0 \rightarrow W/\phi(\tau)(\xi_0)$$

are the linear maps induced by f_0 , and $\phi(\tau)$ respectively. Note that p_0 is an isomorphism, because it can be written as a composition of isomorphisms:

$$Y_{\xi_0}/\xi_0 \rightarrow f_0(Y_{\xi_0}) \rightarrow W/\phi(\tau)(\xi_0) .$$

The diagram (19) shows that

- (a) $r_{f_\tau, \xi_0} = r_{\phi(\tau), \xi_0}$ for every sufficiently small $\tau \in N_0 \setminus \{0\}$,
- (b) $p_{f_\tau, \xi_0} = p_0 + \phi(\tau)_0$ for every sufficiently small $\tau \in N_0 \setminus \{0\}$.

Using formula (16) of Lemma 17 and taking into account (a), (b) we see that, via the canonical isomorphisms

$$\begin{aligned} \det(Y_{\xi_0}) &= \xi_0 \otimes \det(Y_{\xi_0}/\xi_0) , \\ \det(W) &= \phi(\tau)(\xi_0) \otimes \det(W/\phi(\tau)(\xi_0)) , \end{aligned}$$

we have for every sufficiently small $\tau \in N_0 \setminus \{0\}$

$$\begin{aligned} \deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X) &= \text{sign}(\mu_{\xi_0} \circ (r_{\phi(\tau), \xi_0} \otimes \det(p_0 + \phi(\tau)_0))) \\ (20) \quad &= \text{sign}(\mu_{\xi_0} \circ (r_{\phi(\tau), \xi_0} \otimes \det(p_0))) . \end{aligned}$$

For the last equality we used $\lim_{t \rightarrow 0}(p_0 + t\phi(\tau)_0) = p_0$ and the continuity of the determinant. Now we use the canonical isomorphism

$$v : W / f_0(Y_{\xi_0}) \otimes \det(f_0(Y_{\xi_0})) \rightarrow \det(W)$$

and we apply Remark 11 to the subspaces $A := \phi(\tau)(\xi_0)$ (for $\tau \neq 0$), $B := f_0(Y_{\xi_0})$ of W . Therefore let

$$\alpha : \phi(\tau)(\xi_0) \rightarrow W / f_0(Y_{\xi_0}) , \quad \beta : f_0(Y_{\xi_0}) \rightarrow W / \phi(\tau)(\xi_0)$$

be the isomorphisms associated with the direct sum decomposition (18). Using the second statement of Remark 11 we see that the equality (2.3) remains true if we replace $r_{\phi(\tau), \xi_0}$ by

$$\alpha \circ r_{\phi(\tau), \xi_0} : \xi_0 \rightarrow W / f_0(Y_{\xi_0}) ,$$

and p_0 by

$$\beta^{-1} \circ p_0 : Y_{\xi_0} / \xi_0 \rightarrow f_0(Y_{\xi_0}) .$$

But

$$\alpha \circ r_{\phi(\tau), \xi_0} = \bar{\phi}(\tau)|_{\xi_0} : \xi_0 \rightarrow Y_{\xi_0} / \xi_0 , \quad \beta^{-1} \circ p_0 = \bar{f}_0 : Y_{\xi_0} / \xi_0 \rightarrow f_0(Y_{\xi_0}) .$$

Therefore

$$(21) \quad \deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X) = \text{sign}(\mu_{\xi_0} \circ (\bar{\phi}(\tau)|_{\xi_0} \otimes \det(\bar{f}_0))) \quad \forall \tau \in N_0 \setminus \{0\} .$$

Now recall that $\bar{\phi}(\tau)|_{\xi_0} = a_{f_0}(\tau)$ and that

$$a_{f_0}(\tau) \otimes \det(\bar{f}_0) = u_{f_0}(a_{f_0}(\tau)) = \psi_{f_0}(\tau)$$

by the definitions of u_{f_0} and ψ_{f_0} . This proves the claim in the case of an affine map \mathbf{f} .

In order to prove the statement for a general map \mathbf{f} note that the space \mathcal{F}_{f_0} of maps $\mathbf{f} : N_0 \rightarrow \text{Hom}(V, W)$ satisfying the hypotheses of the theorem is a closed affine subspace of the Fréchet space $\mathcal{C}^\infty(N_0, \text{Hom}(V, W))$. Fix a norm on the line N_0 . For a bounded (with respect to the \mathcal{C}^∞ -topology) subset $\mathcal{K} \subset \mathcal{F}_{f_0}$ we can find $\varepsilon > 0$ such that $[f_\tau]_X$ is defined and is a local diffeomorphism at ξ_0 for every $\mathbf{f} \in \mathcal{K}$ and every $\tau \in N_0 \setminus \{0\}$ with $\|\tau\| < \varepsilon$. This shows that the map

$$\mathbf{f} \mapsto \deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X) \quad \text{for small } \tau \in N_0 \setminus \{0\}$$

is locally constant on \mathcal{F}_{f_0} . But this space is connected and contains affine maps.

(3) Note first that it is sufficient to prove the claimed formula for a special map $\mathbf{f} : N_0 \rightarrow \text{Hom}(V, W)$ satisfying the hypothesis of the theorem. This is the case since, for two such maps \mathbf{f}, \mathbf{g} , the points $\mathbf{f}(\tau), \mathbf{g}(\tau)$ belong to the same chamber (see Definition 15) for any sufficiently small $\tau \in N_0 \setminus \{0\}$.

We will construct a special map $\mathbf{f} : N_0 \rightarrow \text{Hom}(V, W)$ which satisfies the hypothesis of the theorem, is affine, and has the following remarkable property:

P. *There exists $\zeta_0 \in \mathbb{P}(W)$ such that subspace $L_0 := f_\tau^{-1}(\zeta_0)$ is independent of*

$\tau \in N_0$ and $\mathbb{P}(L_0)$ is transversal to X at any intersection point.

The existence of such a map solves our problem. Indeed, since X is compact and $\mathbb{P}(L_0)$ is transversal to X at any intersection point, the intersection $F := \mathbb{P}(L_0) \cap X$ is finite. By (1) we know that, for every sufficiently small $\tau \in N_0 \setminus \{0\}$, the map $[f_\tau]_X$ is defined on all of X . On the other hand our transversality condition implies that ζ_0 is a regular value of these maps.

Taking $\tau = 0$ in the first condition of **P** we see that $\xi_0 \in F$. Applying Proposition 4 to $[f_\tau]_X$ for sufficiently small $\tau \in N_0 \setminus \{0\}$, we obtain

$$\deg_{\nu(f_\tau, \mu)}([f_\tau]_X) = \deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X) + \sum_{\xi \in F \setminus \{\xi_0\}} \deg_{\nu(f_\tau, \mu), \xi}([f_\tau]_X) .$$

Since the second term is obviously independent of $\tau \neq 0$ we get

$$\deg_{\nu(f_\tau, \mu)}([f_\tau]_X) - \deg_{\nu(f_{-\tau}, \mu)}([f_{-\tau}]_X) = \deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X) - \deg_{\nu(f_{-\tau}, \mu), \xi_0}([f_{-\tau}]_X) ,$$

so the result follows from (2).

We conclude the proof of the theorem with the construction of an affine map $\mathbf{f} = (f_\tau)_{\tau \in N_0}$ which satisfies the hypothesis of the theorem and has the property **P**.

Put $K_0 := \ker(f_0)$. Since f_0 is an epimorphism, we have $\dim(K_0) = N - n$ and the space \mathcal{L}_{K_0} of $(N - n + 1)$ -dimensional linear subspaces $L \subset V$ with $K_0 \subset L$ can be identified with $\mathbb{P}(W)$ via f_0 . The subset

$$\mathcal{L}_0 := \{L \in \mathcal{L}_{K_0} \mid L \cap Y_{\xi_0} = \xi_0\}$$

is non-empty and Zariski open, hence open and dense in \mathcal{L}_{K_0} . Let $L_0 \in \mathcal{L}_0$ an element which corresponds to a regular value ζ_0 of the projection

$$[f_0]_X : X \setminus \{\xi_0\} \rightarrow \mathbb{P}(W) .$$

The existence of such a point follows from Sard's theorem.

Note that $\mathbb{P}(L_0)$ is transversal to X at any intersection point $\xi \in \mathbb{P}(L_0) \cap X$. Indeed, the condition $L_0 \cap Y_{\xi_0} = \xi_0$ implies that $\mathbb{P}(L_0)$ is transversal to X at ξ_0 , whereas the condition that $\zeta_0 = f_0(L_0)$ is a regular value of $[f_0]_X : X \setminus \{\xi_0\} \rightarrow \mathbb{P}(W)$ implies that $\mathbb{P}(L_0)$ is transversal to X at any point $\xi \in \mathbb{P}(L_0) \cap X \setminus \{\xi_0\}$.

Therefore the obtained $(N - n + 1)$ -dimensional subspace L_0 has the properties:

- (a) $K_0 \subset L_0$,
- (b) $L_0 \cap Y_{\xi_0} = \xi_0$,
- (c) $\mathbb{P}(L_0)$ is transversal to X at any intersection point.

Now we choose a complement for each of the three inclusions in the chain

$$\xi_0 \subset K_0 \subset L_0 \subset V .$$

Let U_0 be an arbitrary complement of ξ_0 in K_0 , l_0 an arbitrary complement of K_0 in L_0 , and M_0 a complement of L_0 in V which is contained in Y_{ξ_0} . The latter complement exists, because by (b) any complement of ξ_0 in Y_{ξ_0} is also a complement of L_0 in V .

We have $\dim(U_0) = N - n - 1$, $\dim(l_0) = 1$, $\dim(M_0) = n - 1$. The sum $l_0 + M_0$ is a complement of K_0 in V , hence f_0 induces an isomorphism $l_0 + M_0 \rightarrow W$. We obtain an induced internal direct sum decomposition of $W = \zeta_0 \oplus W_0$ with

$$\zeta_0 := f_0(l_0) = f_0(L) , \quad W_0 := f_0(M_0) = f_0(Y_{\xi_0}) .$$

With respect to the internal direct sum decompositions

$$V = \xi_0 \oplus U_0 \oplus l_0 \oplus M_0, \quad W = \zeta_0 \oplus W_0$$

the map f_0 is given by a matrix of the form

$$\begin{pmatrix} 0 & 0 & g_0 & 0 \\ 0 & 0 & 0 & h_0 \end{pmatrix},$$

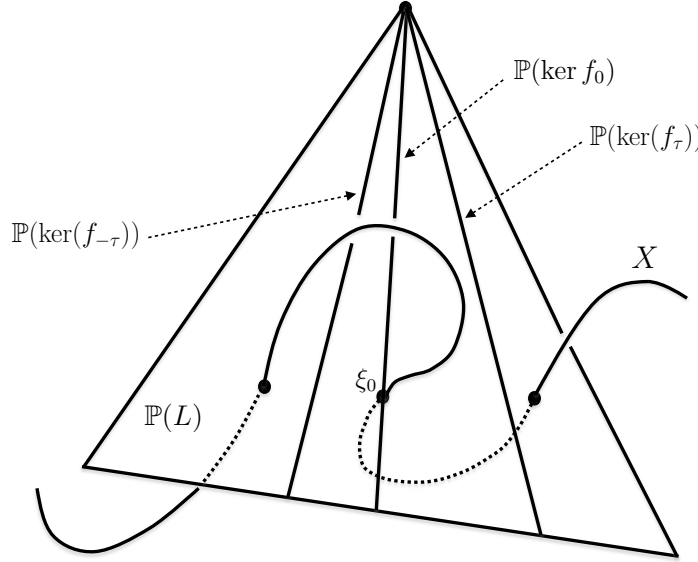
where $g_0 : l_0 \rightarrow \zeta_0$, $h_0 : M_0 \rightarrow W_0$ are the isomorphisms induced by f_0 . We denote by $r_\tau : \xi_0 \rightarrow \zeta_0$ the morphism defined as the image of τ under the composition

$$N_0 \xrightarrow{a_{f_0} \simeq} \text{Hom}(\xi_0, W/f_0(Y_{\xi_0})) = \text{Hom}(\xi_0, W/W_0) \xrightarrow{\simeq} \text{Hom}(\xi_0, \zeta_0),$$

and we define

$$f_\tau := \begin{pmatrix} r_\tau & 0 & g_0 & 0 \\ 0 & 0 & 0 & h_0 \end{pmatrix}.$$

The map $\mathfrak{f} = (f_\tau)_\tau : N_0 \rightarrow \text{Hom}(V, W)$ is affine and satisfies the hypothesis of the theorem (because $A_{f_0} \circ \mathfrak{f}_{*,0}$ coincides with a_{f_0} by definition of r_τ). Moreover, since h_0 is an isomorphism, for every $\tau \in N_0$ the subspace $f_\tau^{-1}(\zeta_0)$ coincides with L_0 . Taking into account (b) we see that \mathfrak{f} satisfies property **P**, which concludes the proof.



■

Corollary 21. *Let $f_0 \in \mathcal{W}_X^0$ be a regular point on the wall and let*

$$\gamma = (g_t)_t : (-\varepsilon, \varepsilon) \rightarrow \text{Hom}(V, W)$$

be a smooth path such that $\gamma(0) = f_0$ and the image $[\dot{\gamma}(0)]$ of the velocity vector in $N_{\mathcal{W}_X^0, f_0}$ is non-zero. Then for every sufficiently small $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$ we have:

- (1) $g_t \in \text{Hom}(V, W)_X$ and $[g_t]_X$ is a local diffeomorphism at ξ_{f_0} ,
- (2) $\deg_{\nu(g_t, \mu), \xi_0}([g_t]_X) = \text{sign}(t) \text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}([\dot{\gamma}(0)])))$,
- (3) $\deg_{\nu(g_t, \mu)}([g_t]_X) - \deg_{\nu(g_{-t}, \mu)}([g_{-t}]_X) = 2 \text{sign}(t) \text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}([\dot{\gamma}(0)])))$.

Proof. We may suppose that γ is given by $g_t = f_{t\tau_0}$, where $\tau_0 \in N_{\mathcal{W}_X^0, f_0} \setminus \{0\}$ and $\mathfrak{f} : N_{\mathcal{W}_X^0, f_0} \rightarrow \text{Hom}(V, W)$ is a map satisfying the hypothesis of Theorem 20. ■

This implies the following general difference formula for paths which cross the wall transversally in regular points:

Theorem 22. (*difference formula*) *Let $\gamma : [0, 1] = (g_t)_{t \in [0, 1]} : [0, 1] \rightarrow \text{Hom}(V, W)$ be a smooth path such that*

- (1) $g_0, g_1 \in \text{Hom}(V, W)_X$,
- (2) $\text{im}(\gamma)$ intersects the wall \mathcal{W}_X only in regular points,
- (3) γ is transversal to \mathcal{W}_X^0 .

Let $\gamma^{-1}(\mathcal{W}_X) = \{t_1, \dots, t_k\}$. Then one has

$$\deg_{\nu(g_1, \mu)}([g_1]_X) - \deg_{\nu(g_0, \mu)}([g_0]_X) = 2 \sum_{i=1}^k \text{sign}(\mu_{\xi_{g_{t_i}}} \circ (\psi_{g_{t_i}}([\dot{\gamma}(t_i)]))) ,$$

where $[\dot{\gamma}(t_i)]$ denotes the projection of the velocity vector $\dot{\gamma}(t_i)$ to the normal line $N_{\mathcal{W}_X^0, g_{t_i}}$

Proof. Suppose $t_1 < \dots < t_k$. The map $t \mapsto \deg_{\nu(g_t, \mu)}([g_t]_X)$ is well defined and constant on each of the intervals

$$[0, t_1) , (t_1, t_2) , \dots , (t_{k-1}, t_k) , (t_k, 1] .$$

The jumps are given by Corollary 21. ■

We will now prove that any two points $f_0, f_1 \in \text{Hom}(V, W)_X$ can be connected by a path intersecting the wall transversally in finitely many regular points. This result, which has important consequences, is based on the following

Theorem 23. *The irregular locus $\mathcal{B}_X = \mathcal{W}_X \setminus \mathcal{W}_X^0$ is closed and the complement $\text{Hom}(V, W) \setminus \mathcal{B}_X$ is connected.*

Proof. We shall identify \mathcal{B}_X with the union of the images of two smooth proper maps

$$\hat{q}_X : \hat{\mathcal{J}}_X \rightarrow \text{Hom}(V, W) , \quad r_X : \mathcal{D}_X \rightarrow \text{Hom}(V, W)$$

of index -2, $n - N - 1$ respectively. Then the result follows from Lemma 5.7 in [Te].

Denote by Δ the diagonal of the product $X \times X$ and consider the real blow up $\widehat{X \times X}_\Delta$ of $X \times X$ along Δ (see [Wh] section 3, and [Po] section 4 for a similar construction). Set theoretically one has

$$\begin{aligned} \widehat{X \times X}_\Delta &= \{(X \times X) \setminus \Delta\} \cup \mathbb{P}(N_\Delta) = \{(X \times X) \setminus \Delta\} \cup \mathbb{P}(T_X) = \\ &\quad \{(X \times X) \setminus \Delta\} \cup \mathbb{P}(Y_X/(\lambda_V|_X)) . \end{aligned}$$

Therefore a point $\zeta \in \mathbb{P}(N_\Delta)$ above a diagonal point $(\xi, \xi) \in \Delta$ defines a plane $\pi_\zeta \subset Y_\xi$ containing ξ . We have a natural *smooth* map

$$\psi : \widehat{X \times X}_\Delta \rightarrow G_2(V)$$

defined in the following way:

$$\psi(\zeta) := \begin{cases} \xi + \eta & \text{when } \zeta = (\xi, \eta) \in (X \times X) \setminus \Delta , \\ \pi_\zeta & \text{when } \zeta \in \mathbb{P}(N_\Delta) . \end{cases}$$

We define

$$\hat{\mathcal{J}}_X := \{(f, K, \zeta) \in \text{Hom}(V, W) \times G_{N-n}(V) \times \widehat{X \times X_\Delta} \mid \psi(\zeta) \subset K \subset \ker(f)\} .$$

This space has a natural structure of a vector bundle of rank n^2 over the incidence variety

$$\hat{\mathcal{I}}_X = \{(\zeta, K) \in \widehat{X \times X_\Delta} \times G_{N-n}(V) \mid \psi(\zeta) \subset K\} .$$

The variety $\hat{\mathcal{I}}_X$ is a locally trivial fibre bundle over $\widehat{X \times X_\Delta}$ with $n(N - n - 2)$ -dimensional fibre, hence smooth of dimension $nN - n^2 - 2$. This shows that $\dim(\hat{\mathcal{J}}_X) = nN - 2$, so the rank of projection $\hat{q}_X : \hat{\mathcal{J}}_X \rightarrow \text{Hom}(V, W)$ is -2 .

Put now

$$\mathcal{D}_X := \{(f, L, \xi) \in \text{Hom}(V, W) \times G_{N-n+1}(V) \times X \mid \xi \subset L \subset \ker(f)\} .$$

\mathcal{D}_X has a natural structure of a $(N + 1)(n - 1)$ -dimensional manifold, because it is a rank $(n - 1)n$ vector bundle over a $n - 1 + (n - 1)(N - n)$ dimensional basis. Let $r_X : \mathcal{D}_X \rightarrow \text{Hom}(V, W)$ be the projection on the first factor. This is a smooth proper map of index $n - N - 1 \leq -2$. On the other hand, taking into account Definition 9 we see that

$$\mathcal{B}_X = \hat{q}_X(\hat{\mathcal{J}}_X) + r_X(\mathcal{D}_X) ,$$

hence

$$\text{Hom}(V, W) \setminus \mathcal{B}_X = (\text{Hom}(V, W) \setminus \hat{q}_X(\hat{\mathcal{J}}_X)) \setminus r_X(\mathcal{D}_X \setminus r_X^{-1}(\hat{q}_X(\hat{\mathcal{J}}_X))) .$$

Applying now Lemma 5.7 in [Te] to the proper morphisms

$$\hat{q}_X , r_X|_{\mathcal{D}_X \setminus r_X^{-1}(\hat{q}_X(\hat{\mathcal{J}}_X))} : \mathcal{D}_X \setminus r_X^{-1}(\hat{q}_X(\hat{\mathcal{J}}_X)) \rightarrow (\text{Hom}(V, W) \setminus \hat{q}_X(\hat{\mathcal{J}}_X))$$

we see that the natural maps

$$\pi_0(\text{Hom}(V, W) \setminus \hat{q}_X(\hat{\mathcal{J}}_X), f_0) \rightarrow \pi_0(\text{Hom}(V, W), f_0) ,$$

$$\pi_0(\text{Hom}(V, W) \setminus \mathcal{B}_X, f_0) \rightarrow \pi_0(\text{Hom}(V, W) \setminus \hat{q}_X(\hat{\mathcal{J}}_X), f_0)$$

are bijections, so that $\text{Hom}(V, W) \setminus \mathcal{B}_X$ is connected, as claimed. \blacksquare

Corollary 24. *Any pair $(f_0, f_1) \in \text{Hom}(V, W)_X \times \text{Hom}(V, W)_X$ can be connected by a smooth path $\gamma : [0, 1] \rightarrow \text{Hom}(V, W)$ which intersects the wall \mathcal{W}_X in finitely many regular points, all intersection points being transversal.*

Proof. Since $\text{Hom}(V, W) \setminus \mathcal{B}_X$ is connected, the pair (f_0, f_1) can be connected by a smooth path $\alpha : [0, 1] \rightarrow \text{Hom}(V, W)$ which intersects the wall only in regular points. Using a well-known transversality principle (see [DoK] p. 143) we find small perturbations of α which coincide with α on a neighbourhood of $\{0, 1\}$ and are transversal to the map $\mathcal{W}_X^0 \hookrightarrow \text{Hom}(V, W)$. \blacksquare

Therefore Corollary 24 states that the difference

$$\deg_{\nu(f_1, \mu)}([f_1]_X) - \deg_{\nu(f_0, \mu)}([f_0]_X)$$

can always be computed using such a path from f_0 to f_1 and the difference formula given by Corollary 22. Combining with Remark 18 one obtains the following important general property of the degree map $\deg_\mu^X : \pi_0(\text{Hom}(V, W) \setminus \mathcal{W}_X) \rightarrow \mathbb{Z}$:

Corollary 25. *Let $\mu : X \times \det(W) \rightarrow \det(Y_X)$ be an orientation parameter. Then:*

- (1) *All values of the degree map $\deg_\mu^X : \pi_0(\text{Hom}(V, W) \setminus \mathcal{W}_X) \rightarrow \mathbb{Z}$ are congruent modulo 2.*

(2) If $a \in \text{im}(\deg_\mu^X)$, then any integer c with $-|a| \leq c \leq |a|$ which has the same parity as a also belongs to $\text{im}(\deg_\mu^X)$.

Proof. The first statement follows directly from the difference formula. For the second statement use Corollary 24 and take into account that

- the jump when crossing the wall transversally at a regular point is ± 2 ,
- the image of \deg_μ^X is invariant under the involution $-\text{id}_\mathbb{Z}$, by Remark 18.

■

2.4. Examples.

2.4.1. *Projecting a hyperquadric.* Let X be the hyperquadric of $\mathbb{P}_\mathbb{R}^n$ ($n \geq 2$) defined by the equation $x_0^2 - \sum_{i=1}^n x_i^2 = 0$. Dehomogenizing with respect to x_0 one gets an obvious identification $X = S^{n-1}$. The relative orientability condition is always satisfied (it is obvious for $n \geq 3$ by Remark 3).

Consider the two projections

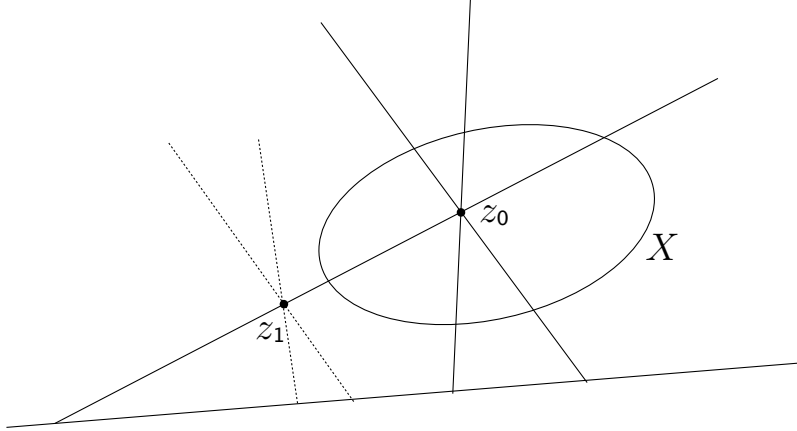
$$[f_0] : \mathbb{P}^n \setminus \{[1, 0, \dots, 0]\} \rightarrow \mathbb{P}^{n-1}, \quad [f_1] : \mathbb{P}^n \setminus \{[0, 0, \dots, 1]\} \rightarrow \mathbb{P}^{n-1}$$

given by

$$[f_0]([x]) := [x_1, \dots, x_n], \quad [f_1]([x]) := [x_0, \dots, x_{n-1}].$$

Via the obvious identification $X = S^{n-1}$, the first map is just the canonical projection $S^{n-1} \rightarrow \mathbb{P}^{n-1}$. The degrees of the the restrictions $[f_0]_X : X \rightarrow \mathbb{P}^{n-1}$, $[f_1]_X : X \rightarrow \mathbb{P}^{n-1}$ with respect to a suitable choice of the trivialization μ are

$$\deg_{\nu(p_0, \mu)}([f_0]_X) = 2, \quad \deg_{\nu(p_1, \mu)}([f_1]_X) = 0.$$



The picture above shows an interesting phenomenon: the projection $[f_1]_X$, which has degree 0, has some fibres consisting of two points, which however come with opposite signs. For instance, the intersection of X with the line connecting z_0 and z_1 consists of two points, which come with the same sign when considered as elements in the fibre of $[f_0]_X$, but with opposite signs when considered as elements in the fibre of $[f_1]_X$.

2.4.2. Degree of rational functions. Let ${}_{\mathbb{R}}\mathbb{F}_n^*$ be the space of pairs of monic polynomials (p, q) with real coefficients of degree n with no common factor. Writing

$$p(t) = t^n + \sum_{i=0}^{n-1} a_i t^i, \quad q(t) = t^n + \sum_{i=0}^{n-1} b_i t^i$$

we see that ${}_{\mathbb{R}}\mathbb{F}_n^*$ can be identified with the Zariski open subset of \mathbb{R}^{2n} defined by the condition $\text{Res}(p, q) \neq 0$.

A pair $(p, q) \in {}_{\mathbb{R}}\mathbb{F}_n^*$ defines a rational function

$$f_{pq}(t) = \frac{p(t)}{q(t)},$$

hence ${}_{\mathbb{R}}\mathbb{F}_n^*$ can be identified with the space of rational functions of degree n sending ∞ to 1. Such a rational function f_{pq} has an extension $F_{pq} : \mathbb{P}_{\mathbb{R}}^1 \rightarrow \mathbb{P}_{\mathbb{R}}^1$. Taking the homogenizations $P \in \mathbb{R}[t_0, t_1]_n$, $Q \in \mathbb{R}[t_0, t_1]_n$ we see that F_{pq} is given in homogeneous coordinates by

$$F_{pq}(t_0, t_1) := [P(t_0, t_1), Q(t_0, t_1)].$$

The set of possible degrees $\deg(F_{pq})$ when (p, q) varies in the space ${}_{\mathbb{R}}\mathbb{F}_n^*$ is known. The result is given by Brockett Theorem (see [Se] p. 41 or [By] Theorem 2.1).

Theorem 26. *The space ${}_{\mathbb{R}}\mathbb{F}_n^*$ has $n + 1$ connected components ${}_{\mathbb{R}}\mathbb{F}_{uv}^*$, where*

$$u + v = n, \quad u \geq 0, \quad v \geq 0$$

and a pair $(p, q) \in {}_{\mathbb{R}}\mathbb{F}_n^$ belongs to ${}_{\mathbb{R}}\mathbb{F}_{uv}^*$ if and only if $\deg(F_{pq}) = u - v$.*

Therefore the set of possible degrees is $-n, -n + 2, \dots, n - 2, n$. Note that the degree of the map $F_{pq}^{\mathbb{C}} : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ defined by a pair $(p, q) \in {}_{\mathbb{R}}\mathbb{F}_n^*$ is always n . We see that in this case *all* values of the real degree allowed by the general estimates and congruences in Proposition 5 can occur.

The pair of polynomials corresponding to the rational function

$$g_{uv}(t) = 1 + \sum_{i=1}^v \frac{-1}{t+i} + \sum_{j=1}^u \frac{1}{t-j}.$$

is an element of ${}_{\mathbb{R}}\mathbb{F}_{uv}^*$.

Note that the map F_{pq} associated with a pair $(p, q) \in {}_{\mathbb{R}}\mathbb{F}_n^*$ can be identified with the composition $[\pi_{pq}] \circ v_n$, where $v_n : \mathbb{P}_{\mathbb{R}}^1 \rightarrow \mathbb{P}_{\mathbb{R}}^n$ is the Veronese map of degree n , and $\pi_{pq} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^2$ is the projection defined by (P, Q) . Therefore the theorem of Brockett identifies $n + 1$ chambers in the complement

$$\text{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^2)_{v_n(\mathbb{P}^1)} = \text{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^2) \setminus \mathcal{W}_{v_n(\mathbb{P}^1)}.$$

3. EXAMPLES

3.1. Conjugation manifolds. Let (X, τ) a topological space endowed with an involution such that $H^{\text{odd}}(X, \mathbb{Z}_2) = 0$. We recall from [HHP] that a cohomology frame for (X, τ) is a pair (κ, σ) , where

- (1) $\kappa : H^{2*}(X, \mathbb{Z}_2) \rightarrow H^*(X^{\tau}, \mathbb{Z}_2)$ is a group isomorphism.
- (2) $\sigma : H^{2*}(X, \mathbb{Z}_2) \rightarrow H_{\mathbb{Z}_2}^{2*}(X, \mathbb{Z}_2)$ is a group morphism which is a section of the restriction map $\rho : H_{\mathbb{Z}_2}^{2*}(X, \mathbb{Z}_2) \rightarrow H^{2*}(X, \mathbb{Z}_2)$,

such that the following conjugation equation holds:

$$r \circ \sigma(a) = \kappa(a)u^m + q(a) \quad \forall a \in H^{2m}(X, \mathbb{Z}_2) .$$

Here r denotes the restriction map $H_{\mathbb{Z}_2}^*(X, \mathbb{Z}_2) \rightarrow H_{\mathbb{Z}_2}^*(X^\tau, \mathbb{Z}_2) = H^*(X, \mathbb{Z}_2)[u]$, and $q(a)$ is an element in $H^*(X, \mathbb{Z}_2)[u]$ whose degree with respect to u is smaller than m .

If a cohomology frame for (X, τ) exists, then it is unique, natural with respect to equivariant maps, and κ, σ are automatically ring isomorphisms (not only group isomorphisms).

A \mathbb{Z}_2 -space (X, τ) which admits a cohomology frame is called *conjugation space*. Examples of conjugation spaces are: all complex Grassmann manifolds (with the standard Real structure), all toric manifolds (with their standard Real structure).

Let now (X, τ) be a paracompact conjugation space with cohomology frame (κ, σ) , and let $(\tilde{E}, \tilde{\tau})$ a Real complex vector bundle on X . Denote by $\bar{c}(E)$ the image of the total Chern class $c(E)$ in $H^*(X, \mathbb{Z}_2)$. Then $\kappa(\bar{c}(E)) = w(E^{\tilde{\tau}})$ (see [HHP] p. 950).

Proposition 27. *Let $(X, \tau), (Y, \iota)$ be Real complex manifolds which are conjugation spaces with respect to their real structures, and let $f : X \rightarrow Y$ be a Real holomorphic map. Then f is relatively orientable if and only if $f^*(c_1(Y)) \equiv c_1(X) \pmod{2}$.*

Proof. One has

$$f(\mathbb{R})^* w_1(T_{Y(\mathbb{R})}) = f(\mathbb{R})^*(\kappa_Y \bar{c}_1(T_Y)) = \kappa_X(f^*(\bar{c}_1(T_Y))) = \kappa_X(c_1(T_X)) = w_1(T_{X(\mathbb{R})}),$$

where $(\kappa_X, \sigma_X), (\kappa_Y, \sigma_Y)$ are the cohomology frames of (X, τ) and (Y, ι) respectively, T_X, T_Y the two tangent bundles regarded as complex vector bundles, $X(\mathbb{R}), Y(\mathbb{R})$ the fixed point loci, and $f(\mathbb{R})$ the map $X(\mathbb{R}) \rightarrow Y(\mathbb{R})$ induced by f . It suffices to apply Remark 2. \blacksquare

3.2. Plücker embeddings of real Grassmann manifolds. Let V_0 be real vector space of dimension $p + q$, $G_q(V_0)$ the Grassmann manifold of q -planes in V_0 . Take $V = \wedge^q V_0$ and let X be the image of the Plücker embedding $Pl : G_q(V_0) \rightarrow \mathbb{P}(V)$. Denoting by U the tautological rank q bundle on $G_q(V_0)$ we have a natural identification $T_{G_q(V_0)} = \text{Hom}(U, V_0/U)$, which shows that

$$w_1(G_q(V_0)) = (pq + 1)w_1(U) .$$

On the other hand it is well-known that $Pl^*(\lambda_V) = \det(U)$. In our case we have $n = m + 1 = pq + 1$. Since on any Grassmann manifold one has $w_1(U) \neq 0$, we have

Corollary 28. *A map $[f]_X : Pl(G_q(V_0)) \rightarrow \mathbb{P}(W)$ associated with a linear map $f \in \text{Hom}(\wedge^q(V_0), W)$ satisfying $\ker(f) \cap Pl(G_q(V_0)) = \emptyset$ is relatively orientable if and only if $pq + 1 \equiv p + q \pmod{2}$, i.e., iff p and q are not both even.*

Consider now the special case when $V_0 = S^{p+q-1}W_0^\vee$, where $W_0 = \mathbb{R}^2$, and take $W := S^{pq}W_0^\vee$. This special case is important because we have a *standard* linear epimorphism

$$\varphi_{\text{Wronski}} : \wedge^q(S^{p+q-1}W_0^\vee) \rightarrow S^{pq}W_0^\vee, \quad \varphi_{\text{Wronski}}(F_1 \wedge \cdots \wedge F_q) := W(F_1, \dots, F_q),$$

where

$$W : [S^{p+q-1}(W_0^\vee)]^q \rightarrow S^{pq}W_0^\vee$$

denotes the homogeneous Wronskian. Alternatively, φ_{Wronski} can be defined as the composition of the *standard isomorphism*

$$\wedge^q(S^{p+q-1}W_0^\vee) \xrightarrow{\simeq} S^q(S^pW_0^\vee)$$

with the natural projection $S^q(S^pW_0^\vee) \rightarrow S^{pq}W_0^\vee$ (see [AC]).

Up to a constant factor the map φ_{Wronski} can also be obtained using the inhomogeneous Wronskian

$$w : (\mathbb{R}[s]_{\leq p+q-1})^q \rightarrow \mathbb{R}[s]_{\leq pq} ,$$

via the obvious identifications $\mathbb{R}[s]_{\leq k} \simeq S^k(W_0^\vee)$ (see [AC] section 2.8). With this remark the results of [EG1] (where the inhomogeneous Wronskian is used) apply. First, it is well known that $\mathbb{P}(\ker(\varphi_{\text{Wronski}})) \cap G_q(S^{p+q-1}W_0^\vee) = \emptyset$, so we have a well-defined projection

$$[\varphi_{\text{Wronski}}] : G_q(S^{p+q-1}W_0^\vee) \rightarrow \mathbb{P}(S^{pq}W_0^\vee)$$

for which the degree is known (see [EG1]).

When p and q are not both even, then $[\varphi_{\text{Wronski}}]$ is relatively orientable and:

$$|\deg|[\varphi_{\text{Wronski}}] = \begin{cases} 0 & \text{if } p, q \text{ are both odd} \\ I(p, q) & \text{if } p+q \text{ is odd} . \end{cases}$$

Here $I(\cdot, \cdot)$ is symmetric, and for $2 \leq p \leq q$ the integer $I(p, q)$ is given by:

$$I(p, q) = \frac{1!2! \cdots (p-1)!(q-1)!(q-2)! \cdots (q-(p-1))! \left(\frac{pq}{2}\right)!}{(q-p+2)!(q-p+4)! \cdots (q+p-2)! \left(\frac{q-p+1}{2}\right)! \left(\frac{q-p+3}{2}\right)! \cdots \left(\frac{q+p-1}{2}\right)!} .$$

Remark 29. Using the first formula in Lemma 12 and Lemma 15 in [OT2] one obtains a canonical isomorphism

$$\det(T_{Pl(G_q(S^{p+q-1}W_0^\vee))}^\vee) \otimes [f]^*(\det(T_{\mathbb{P}(S^{pq}W_0^\vee)})) =$$

$$[\det(W_0)^\vee]^{\otimes \frac{q(q-1)(p^2-2p-q)}{2}} \otimes [\det(U)^\vee]^{\otimes (p-1)(q-1)} ,$$

for any $f \in \text{Hom}(\wedge^q(S^{p+q-1}W_0^\vee), S^{pq}W_0^\vee)$ with $\mathbb{P}(\ker(f)) \cap Pl(G_q(S^{p+q-1}W_0^\vee)) = \emptyset$. This shows that $[f]_{Pl(G_q(S^{p+q-1}W_0^\vee))}$ is canonically relatively oriented if and only if either $p, q \in 2\mathbb{N} + 1$, or $p \in 2\mathbb{N} + 1$ and $q \in 4\mathbb{N}$, or $p \in 2\mathbb{N}$ and $q \in 4\mathbb{N} + 1$.

Therefore, if the pair (p, q) satisfies one of these three conditions then, for any regular value $[P] \in \mathbb{P}(S^{pq}W_0^\vee)$ of the Wronski map $[\varphi_{\text{Wronski}}]$, one can associate an intrinsic sign to any element in the fibre $[\varphi_{\text{Wronski}}]^{-1}([P])$ (without having to orient the plane W_0). It would be interesting to have a geometric interpretation of these intrinsic signs.

3.3. Universal pole placement map. Let W_0 be a real plane, \underline{V}_0 the trivial bundle $\mathbb{P}(W_0) \times V_0$ on the projective line $\mathbb{P}(W_0)$ with fibre a $p+q$ dimensional vector space V_0 . For $\nu \in \mathbb{N}$ denote by $\text{Quot}_{\mathbb{P}(W_0)}^{p, \nu}(\underline{V}_0)$ the quot space of equivalence classes of quotients $s : \underline{V}_0 \rightarrow Q$ of \underline{V}_0 with $\text{rk}(\ker(s)) = p$ and $\det(\ker(s)) \simeq \mathcal{O}_{\mathbb{P}(W_0)}(-\nu)$. Every such quotient s defines an element $QPl(s) \in \mathbb{P}(\wedge^p V_0 \otimes S^\nu W_0^\vee)$ by the formula

$$QPl(s) := [\wedge^p k_s] .$$

Here $k_s \in \text{Hom}(\ker(s), \underline{V}_0)$ denotes the embedding of $\ker(s)$ in \underline{V}_0 , and

$$\wedge^p k_s \in H^0(\text{Hom}(\wedge^p \ker(s), \wedge^p \underline{V}_0)) = H^0(\wedge^p \underline{V}_0 \otimes (\wedge^p \ker(s))^\vee)$$

is the p -th exterior power of k_s . Since $\det(\ker(s))^\vee \simeq \mathcal{O}_{\mathbb{P}(W_0)}(\nu)$, the section $\wedge^p k_s$ of $\wedge^p \underline{V}_0 \otimes (\wedge^p \ker(s))^\vee$ defines an element of $H^0(\wedge^p \underline{V}_0(\nu)) = \wedge^p V_0 \otimes S^\nu W_0^\vee$, which

is well defined up to multiplication with a non-vanishing scalar. Recall that we have a perfect paring $\wedge^q V_0 \times \wedge^p V_0 \rightarrow \det V_0$, which induces an isomorphism

$$\wedge^p V_0 \rightarrow (\wedge^q V_0)^\vee \otimes \det(V_0) = \text{Hom}(\wedge^q V_0, \det(V_0)) .$$

Therefore the element $QPl(s) = [\wedge^p k_s]$ can be regarded as an element of

$$\mathbb{P}(\text{Hom}(\wedge^q V_0, S^\nu W_0^\vee \otimes \det(V_0))) = \mathbb{P}(\text{Hom}(\wedge^q V_0, S^\nu W_0^\vee))$$

as claimed. Note that in general the map

$$QPl : \text{Quot}_{\mathbb{P}(W_0)}^{p,\nu}(\underline{V}_0) \rightarrow \mathbb{P}(\text{Hom}(\wedge^q V_0, S^\nu W_0^\vee))$$

is not an embedding.

A quotient $[s] \in \text{Quot}_{\mathbb{P}(W_0)}^{p,\nu}(\underline{V}_0)$ defines a central projection

$$\psi_{[s]} : \mathbb{P}(\wedge^q V_0) \setminus \mathbb{P}(\ker(\wedge^p k_s)) \rightarrow \mathbb{P}(S^\nu W_0^\vee) .$$

Since $\mathbb{P}(\wedge^q V_0)$ contains the image of the Plücker embedding

$$Pl : G_q(V_0) \rightarrow \mathbb{P}(\wedge^q V_0) ,$$

it is interesting to study the composition

$$\phi_{[s]} := \psi_{[s]} \circ Pl : G_q(V_0) \setminus \mathbb{P}(\ker(\wedge^p k_s)) \rightarrow \mathbb{P}(S^\nu W_0^\vee)$$

associated with an element $[s] \in \text{Quot}_{\mathbb{P}(W_0)}^{p,\nu}(\underline{V}_0)$. When $\nu = pq$ we can ask if this projection is defined on the all of $G_q(V_0)$, and when p and q are not both even, so that $\phi_{[s]}$ is relatively orientable, one can also ask for the degree.

Remark 30. Consider the special case where $V_0 = S^{p+q-1}W_0^\vee$ and $W_0 = \mathbb{R}^2$. It seems to be well known [EG3] that in this case there exists an element

$$s_{\text{Wronski}} \in \text{Quot}_{\mathbb{P}(W_0)}^{p,pq}(\underline{V}_0)$$

such that

$$\varphi_{[s_{\text{Wronski}}]} = [\varphi_{\text{Wronski}}]$$

is the Wronski projection introduced in section 3.2.

Note that the chamber structure of $\text{Hom}(\wedge^q V_0, S^{pq}W_0^\vee)_{G_q(V_0)}$ induces a chamber structure on the complement

$$\text{Quot}_{\mathbb{P}(W_0)}^{p,pq}(\underline{V}_0) \setminus QPl^{-1}(\mathbb{P}(\mathcal{W}_{G_q(V_0)}))$$

of the pull-back of the projectivized wall $\mathbb{P}(\mathcal{W}_{G_q(V_0)})$ via QPl .

The image $\phi_{[s]}(U) \in \mathbb{P}(S^{pq}W_0^\vee)$ of a q -plane $U \in G_q(V_0) \setminus \mathbb{P}(\ker(\wedge^p k_s))$ can be explicitly described as follows: Denote by $j_U : \underline{U} \rightarrow \underline{V}_0$ the obvious embedding of the trivial rank q -bundle $\underline{U} := \mathbb{P}(W_0) \times U$ in \underline{V}_0 , and by $\rho_U : \underline{V}_0 \rightarrow \underline{V}_0/\underline{U}$ the projection onto the quotient bundle. The determinant $\det(\rho_U \circ k_s)$ of the composition $\rho_U \circ k_s : \ker(s) \rightarrow \underline{V}_0/\underline{U}$ can be regarded as an element of $\wedge^p(V_0/U) \otimes S^{pq}W_0^\vee$. If this element is non-zero it defines an element

$$PP[s](U) \in \mathbb{P}(\wedge^p(V_0/U) \otimes S^{pq}W_0^\vee) = \mathbb{P}(S^{pq}W_0^\vee) ,$$

called the *pole placement* of $[s] \in \text{Quot}_{\mathbb{P}(W_0)}^{p,pq}(\underline{V}_0)$ at $U \in G_q(V_0)$. On the other hand, one can easily prove that $\det(\rho_U \circ k_s)$ is non-zero if and only if $\phi_{[s]}$ is defined at U .

Lemma 31. When $\phi_{[s]}$ is defined at U , one has

$$(22) \quad \phi_{[s]}(U) = PP[s](U) .$$

Proof. Consider the rank p vector bundle K_s on $\mathbb{P}(W_0)$ associated with the sheaf $\ker(s)$, choose $x \in \mathbb{P}(W_0)$ and vectors $l_1, \dots, l_p \in K_{s,x}$. Let $(v_1, \dots, v_p, v_{p+1}, \dots, v_{p+q})$ be a basis of V_0 such that $(v_{p+1}, \dots, v_{p+q})$ is a basis of U . Let $U^\perp := \langle v_1, \dots, v_p \rangle$ and \bar{k}_s the composition of k_s with the projection on U^\perp . Regarding $\wedge^p k_s$ as an element in

$$H^0(\mathbb{P}(W_0), \mathcal{H}om(\det(K_s), \mathcal{H}om(\wedge^q \underline{V}_0, \det(\underline{V}_0))) = \text{Hom}(\wedge^q V_0, \wedge^{p+q} V_0) \otimes S^\nu W_0^\vee$$

we have

$$\begin{aligned} (\wedge^p k_s)(l_1 \wedge \dots \wedge l_p)(v_{p+1} \wedge \dots \wedge v_{p+q}) &= k_s(l_1) \wedge \dots \wedge k_s(l_p) \wedge v_{p+1} \wedge \dots \wedge v_{p+q} = \\ &= \bar{k}_s(l_1) \wedge \dots \wedge \bar{k}_s(l_p) \wedge v_{p+1} \wedge \dots \wedge v_{p+q} = [\det(\rho_U \circ k_s)(l_1 \wedge \dots \wedge l_p)] \otimes (v_{p+1} \wedge \dots \wedge v_{p+q}), \end{aligned}$$

where in the last equality we have used the canonical isomorphism $\det(V_0) = \det(V_0/U) \otimes \det(U)$. Therefore one obtains the following equality in the space $S^\nu W_0^\vee \otimes \det(V_0)$

$$\wedge^p k_s(v_{p+1} \wedge \dots \wedge v_{p+q}) = \det(\rho_U \circ k_s) \otimes (v_{p+1} \wedge \dots \wedge v_{p+q}),$$

which shows that $\wedge^p k_s(v_{p+1} \wedge \dots \wedge v_{p+q})$ and $\det(\rho_U \circ k_s)$ define the same element in $\mathbb{P}(S^\nu W_0^\vee)$. \blacksquare

For a quotient $[s] \in \text{Quot}_{\mathbb{P}(W_0)}^{p,pq}(\underline{V}_0)$ the assignment $U \mapsto PP[s](U)$ defines a rational map $G_q(V_0) \dashrightarrow \mathbb{P}(S^{pq} W_0^\vee)$. We denote by $\text{Rat}(G_q(V_0), \mathbb{P}(S^{pq} W_0^\vee))$ the set of such rational maps. Letting $[s]$ vary in $\text{Quot}_{\mathbb{P}(W_0)}^{p,pq}(\underline{V}_0)$ we obtain a map

$$PP : \text{Quot}_{\mathbb{P}(W_0)}^{p,pq} \rightarrow \text{Rat}(G_q(V_0), \mathbb{P}(S^{pq} W_0^\vee)),$$

which we call the universal pole placement map. Lemma 31 shows that the following diagram is commutative:

$$\begin{array}{ccc} \text{Quot}_{\mathbb{P}(W_0)}^{p,pq}(\underline{V}_0) & \xrightarrow{QPl} & \mathbb{P}(\text{Hom}(\wedge^q V_0, S^{pq} W_0^\vee)) \\ \downarrow PP & & \swarrow Pl^* \\ \text{Rat}(G_q(V_0), \mathbb{P}(S^{pq} W_0^\vee)) & & \end{array}$$

3.4. A real subspace problem. Let W_0, V_0 be real vector spaces of dimensions 2 and $p+q$ respectively. We denote by $S_0^{pq}(\mathbb{P}(W_0))$ the subset of elements \mathfrak{s} in the symmetric power $S^{pq}(\mathbb{P}(W_0))$ consisting of pairwise distinct points.

Subspace problem: Fix a regular algebraic map $\gamma : \mathbb{P}(W_0) \rightarrow G_p(V_0)$ of algebraic degree pq , and an element $\mathfrak{s} \in S_0^{pq}(\mathbb{P}(W_0))$. Count the q -dimensional linear subspaces $U \subset V_0$ such that

$$\mathbb{P}(U) \cap \mathbb{P}(\gamma(\xi)) \neq \emptyset \quad \forall \xi \in \mathfrak{s}. \quad (S_{\gamma, \mathfrak{s}})$$

We will show that this problem has an interesting interpretation which, for general $\mathfrak{s} \in S^{pq}(\mathbb{P}(W_0))$, allows one to associate a sign to every q -dimensional subspace U satisfying the condition $(S_{\gamma, \mathfrak{s}})$ and to compute the total number of solutions of

$(S_{\gamma, \mathfrak{s}})$ when these signs are taken into account. Indeed, for any regular algebraic map $\gamma : \mathbb{P}(W_0) \rightarrow G_p(V_0)$ there exists a *bundle epimorphism*

$$s_\gamma : \underline{V}_0 \rightarrow Q_\gamma$$

classifying γ , i.e., such that

$$\gamma(\xi) = \ker(s_{\gamma, \xi}) \quad \forall \xi \in \mathbb{P}(W_0) .$$

The equivalence class $[s_\gamma] \in \text{Quot}_{\mathbb{P}(W_0)}^{p, pq}(\underline{V}_0)$ is well defined, and the assignment $\gamma \mapsto [s_\gamma]$ defines an embedding

$$\text{Mor}^{pq}(\mathbb{P}(W_0), G_p(V_0)) \rightarrow \text{Quot}_{\mathbb{P}(W_0)}^{p, pq}(\underline{V}_0) ,$$

where $\text{Mor}^{pq}(\mathbb{P}(W_0), G_p(V_0))$ is the space of regular algebraic morphisms $\mathbb{P}(W_0) \rightarrow G_p(V_0)$ of algebraic degree pq . Consider the pole placement map $\phi_{[s_\gamma]} : G_q(V_0) \rightarrow \mathbb{P}(S^{pq}W_0^\vee)$ corresponding to the quotient $[s_\gamma]$.

Let $P_{\mathfrak{s}} \in \mathbb{R}[W_0]_{pq} = S^{pq}W_0^\vee$ be a homogeneous polynomial of degree pq on W_0 whose set of roots coincides with \mathfrak{s} . This polynomial is determined by \mathfrak{s} up to multiplication by a non-vanishing constant.

Lemma 32. *The set of solutions of the problem $(S_{\gamma, \mathfrak{s}})$ can be identified with the fibre $\phi_{[s_\gamma]}$ over $[P_{\mathfrak{s}}]$:*

$$\phi_{[s_\gamma]}^{-1}([P_{\mathfrak{s}}]) = \{U \in G_q(V_0) \mid \mathbb{P}(U) \cap \mathbb{P}(\gamma(\xi)) \neq \emptyset \quad \forall \xi \in \mathfrak{s}\} .$$

Proof. Indeed, using the definition of $\phi_{[s]}$, it follows that for any $U \in G_q(V_0)$ the set of zeros of a polynomial $P_U \in \mathbb{R}[W_0]_{pq}$ representing $\phi_{[s_\gamma]}(U) \in \mathbb{P}(S^{pq}W_0^\vee)$ coincides with the set of points $\xi \in \mathbb{P}(W_0)$ for which $\det(\rho_U \circ k_{s_\gamma, \xi}) \neq 0$, i.e., with the set of points $\xi \in \mathbb{P}(W_0)$ for which $\gamma(x) \cap U \neq \{0\}$. If the latter set is \mathfrak{s} (which has maximal cardinal pq) this condition is equivalent to $[P_U] = [P_{\mathfrak{s}}]$. Therefore $\phi_{[s_\gamma]}(U) = [P_{\mathfrak{s}}]$ if and only if U satisfies the condition $(S_{\gamma, \mathfrak{s}})$. ■

Corollary 33. *Suppose that p and q are not both even and let $\gamma : \mathbb{P}(W_0) \rightarrow G_p(V_0)$ be a regular algebraic map of algebraic degree pq such that*

$$[s_\gamma] \in \text{Quot}_{\mathbb{P}(W_0)}^{p, pq}(\underline{V}_0) \setminus QPl^{-1}(\mathbb{P}(W_{G_q(V_0)})) .$$

Then the pole placement map $\phi_{[s_\gamma]} : G_q(V_0) \rightarrow \mathbb{P}(S^{pq}W_0^\vee)$ is well defined. Choose a relative orientation ν of $\phi_{[s_\gamma]}$. There exists an open dense subset $\mathfrak{S}_\gamma \subset S_0^{pq}(\mathbb{P}(W_0))$ such that for any $\mathfrak{s} \in \mathfrak{S}_\gamma$ one has

$$\sum_{U \text{ solves } (S_{\gamma, \mathfrak{s}})} \varepsilon_{U, \nu} = \deg_\nu(\phi_{[s_\gamma]}) .$$

Proof. The map $\mathfrak{s} \mapsto [P_{\mathfrak{s}}]$ identifies $S_0^{pq}(\mathbb{P}(W_0))$ with an open subset of $\mathbb{R}[W_0]_{pq} = S^{pq}W_0^\vee$. It suffices to apply Sard theorem to the map $\phi_{[s_\gamma]}$ and to take into account that the set of regular values of a proper smooth map is always open. ■

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Christian Okonek:

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 150, CH-8057 Zürich,
e-mail: okonek@math.uzh.ch

Andrei Teleman:

CMI, Aix-Marseille Université, LATP, 39 Rue F. Joliot-Curie, 13453 Marseille Cedex 13,
e-mail: teleman@cmi.univ-mrs.fr